Universidade de Aveiro Departamento de Matemática
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2014

Carla David Reis

Topologia via categorias enriquecidas

Topology via enriched categories

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Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, Programa Doutoral em Matemática e Aplicações da Universidade de Aveiro e Universidade do Minho, realizada sob a orientação científica do Doutor Dirk Hofmann, Professor Auxiliar do Departamento de Matemática da Universidade de Aveiro

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Doutor Gavin Jay Seal
Lecturer, Section de Mathématiques, faculte des Sciences de Base, École Polythecnique Fédérale de Lausanne, Suíça

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palavras-chave

Categoria enriquecida num quantal, V-categoria, V-distribuidor, adjunção, completude, injectividade, exponenciabilidade, conjunto ordenado, espaço métrico, espaço métrico probabilístico

resumo

Tendo como ponto de partida a caracterização de espaços métricos probabilísticos como categorias enriquecidas no quantal Δ, estabelecemos condições que permitem a generalização de resultados que relacionam sucessões de Cauchy, convergência de sucessões, adjunções de V-distribuidores e a sua representabilidade. Também estabelecemos a equivalência entre L-injectividade e L-completude. Caracteriza-se L-completude via a imersão de Yoneda, e injectividade é relacionada com exponenciabilidade. Considera-se outra forma de completude e analisa-se o modelo das bolas formais.

keywords

Category enriched in a quantale, V-category, V-distributor, adjunction, completeness, injectivity, exponentiability, ordered set, metric space, probabilistic metric space

abstract

Having as a starting point the characterization of probabilistic metric spaces as enriched categories over the quantale Δ, conditions that allow the generalization of results relating Cauchy sequences, convergence of sequences, adjunctions of V-distributors and its representability are established. Equivalence between L-completeness and L-injectivity is also established. L-completeness is characterized via the Yoneda embedding, and injectivity is related with exponentiability. Another kind of completeness is considered and the formal ball model is analyzed.

Contents

Notation

$X, Y, Z, ...$	sets
$x, y, z, ...$	elements of a set
$V, W, Q, ...$	quantales
$u, v, w, ...$	elements of a quantale
$f, g, h, ...$	maps, V-functors, elements of Δ
$r, s, t, ...$	relations and V-relations
$a, b, c, ...$	structure of a V-category
$\varphi, \psi, ...$	distributors and V-distributors
$m, n, p, ...$	elements of $[0, +\infty]$
$\delta, \epsilon, ...$	elements of $[0, 1]$
$F, G, H, ...$	functors

Introduction

Traditionally, Category Theory is considered an "organising theory" with (almost) every mathematical discipline as a particular instance. However, it soon became clear that not only the collection of mathematical objects such as topological spaces or groups form a category, but also mathematical structures themselves may be seen as examples of the definition of a category. In fact, in 1973 F.W. Lawvere [Law73] wrote a a truly innovative article where he considers the points of a (generalised) metric space X as the objects of a category X and lets the distance

$$d(x,y) \in [0,\infty]$$

play the role of the hom-set of x and y. In general, given a monoidal closed category $(\mathsf{V},\otimes,k)$, a V-category in the sense of [EK, Kel82] is a pair $(X, a : X \times X \to \mathsf{V})$ together with

$$k \to a(x,x) \qquad \text{and} \qquad a(x,y) \otimes a(y,z) \to a(x,z)$$

satisfying the identity and associative laws. However, the theory of categories enriched in a genuine symmetric monoidal closed category can become quickly technically very demanding, which prompted many authors to restrict themselves to the case where the enrichment takes place in a quantale, that is, in a thin symmetric monoidal closed category where "all diagrams commute" and, therefore, all coherence issues disappear. This way, the employed categorical notions and techniques have a very elementary formulation, however the theory still includes many interesting examples such as ordered sets, metric spaces and probabilistic metric spaces. We refer to the work [BvBR98, Rut98, Kop88, Fla91, Fla97, FSW96, FK97], and stress, in particular, the work of Flagg on continuity spaces (categories enriched in a *value quantale*).

In this thesis we use the framework of categories enriched in a quantale to study properties of objects such as (probabilistic) metric spaces. Hence, our work is primarily based on the notions of distributor and adjunction, and their conceptual power allowed us to establish results that generalise the ones achieved by Lawvere for metric spaces.

This is the case of Cauchy sequences, which is proved to be equivalent, under certain conditions, to adjunctions of distributors, and of the convergence of Cauchy sequences that is equivalent to the representability of the corresponding adjunction. Some of these results had been previously obtained in [Fla97], but now it is shown that they are still valid under less demanding conditions. Furthermore, we studied function spaces of categories enriched in a quantale, verifying, in particular, that injective V-categories are exponentiable. References regarding work on this subject include [CH06, CH09, CHS09, HR13].

Throughout this work we pursue to use examples to illustrate this approach. The characterization of ordered sets and metric spaces using enriched category theory is well known ([Law73, Woo04, BvBR98, CHT04, Kop88, Rut98]), but we were specially interested in the example of probabilistic metric spaces that are categories enriched in the quantale Δ, of distribution functions. Regarding this subject we refer to the work of [Fla97, GR02, Cha09, KM75, HR13, GV94], but also to the "classic" sources such as [Men42, SS83].

The definition of probabilistic metric space is very close to the one of fuzzy metric space as in [GV94], however, there, a distribution function is required to be continuous, not only left continuous. This slight difference raises some important discrepancies, in particular Δ is not cocomplete if its elements are continuous distribution functions, but it is cocomplete if we just demand left continuity. In [KM75] fuzzy metric spaces were also studied considering, however, the same definition as for probabilistic metric spaces. The first L-completion of a probabilistic metric space appeared in [She66] and in [Cha09] it is concluded that a probabilistic metric space is L-complete if it is Cauchy complete.

We start by reviewing concepts and results of order theory expressed in categorical language, making an early description of ordered sets as categories enriched in the quantale $\mathsf{2} = \{\mathsf{true}, \mathsf{false}\}$. The categories Rel, Ord and Dist are introduced, the concept of complete ordered set is presented as an adjunction of distributors and the notion of completely distributive quantale is characterised. It is also in this first part that we study (lax) morphisms of quantales, establishing relationships between the most relevant quantales for our approach. The set of distribution functions, Δ, is introduced and it is proven that it is a completely distributive quantale.

The second chapter focuses on the theory of categories enriched in a quantale. The aim of this chapter is to recall the fundamental concepts, in particular those of V-relation, V-category, V-functor and V-distributor, and characterise the most relevant categories

(V-Rel, V-Cat, V-Dist). It is shown that 2-Cat $\simeq$ Ord, $[0, +\infty]$-Cat $\simeq$ Met and that Δ-Cat $\simeq$ ProbMet. Furthermore, it is verified that morphisms of quantales induce (lax) functors between the categories mentioned before. From the analysis of the main examples, it is concluded that there is a relevant relation between Met and ProbMet translated by adjunctions of (lax) functors between these categories.

The main goal of the third chapter is the generalisation of some conclusions reached in [Law73], in the context of metric spaces, for any V-category. In particular, a L-complete V-category is defined as a V-category in which every adjunction of V-distributors $\varphi \dashv \psi : X \longrightarrow\!\!\!\!\!\!\circ\;\; 1$ is representable, and it is proven that, under some conditions, this concept is equivalent to the classic notion based on Cauchy sequences. To this end, a closure operator (not necessarily a topology) in a V-category is considered, the conditions which make a Cauchy sequence correspond to a pair of adjoint V-distributors are established, and it is proved that, if the neutral element of the quantale is the top element, such sequence converges to x if and only if the adjunction is representable by x. Other characterisations of L-complete V-categories are also made through the functors $(-)_*$ and $(-)^*$ and the Yoneda functor, as well as through the concept of L-injectivity, proving that they are equivalent notions. Furthermore, we analyse the interactions between the concepts discussed and functors induced by morphisms of quantales, concluding that they preserve Cauchy sequences and convergence, and the conditions for such a functor to preserve and reflect L-complete V categories are established. Finally, we make a comparison with the work of R. C. Flagg [Fla97] and V. Gregori and S. Romaguera [GR02] concluding that the results obtained are equivalent. However, the most important aspect which we must emphasise at this point, is that there was no need to work in a topological space and that the set of conditions required to obtain the same results is, in this work, less demanding than the one assumed in [Fla97]. Furthermore, it is considered that the use of categorical language is more suitable and clear, in this context, than the analytical notation of Flagg.

In the last chapter, we relate the concepts of injectivity and exponentiability in the context of V-categories. It is proved that, under certain conditions, injective V-categories are exponentiable, in particular any quantale V viewed as a V-category is exponentiable. In such conditions, the full subcategory V-InjCat of V-Cat, whose objects are injectives V-categories, is Cartesian closed. Taking as inspiration the formal ball model studied by [KW11] and the context developed in section 2 of [CH09], a suitable class Φ of right adjoint V-distributors is considered and the respective concepts of Φ-injectivity and of Φ-completeness built, concluding that they are equivalent. The relevance of this analysis is demonstrated by the verification that the formal ball model is Φ-complete but not complete.

Chapter 1

Ordered Sets

The theory of ordered sets has been extensively studied from the viewpoint of categories enriched in the quantale $2 = \{\mathsf{true}, \mathsf{false}\}$, and concepts and results were described using this language. In this chapter we recall the most relevant, as a motivation for characterising categories enriched in any quantale. An excellent reference on this subject is [Woo04], but also [Tho08].

1.1 The ordered category of relations

Given sets X and Y, a *relation from* X *to* Y is a subset of the Cartesian product $X \times Y$, denoted by $X \looparrowright Y$. In certain contexts, a relation r from X to Y can also be described as a function r from the Cartesian product of X and Y to the set of true values $2 = \{\mathsf{true}, \mathsf{false}\}$ or as a function from X to PY, that identifies the elements of Y that are related to an element of X. Hence, relations of the type $1 \looparrowright X$ and $X \looparrowright 1$ can be understood as subsets of PX.
Any function f can be seen as a relation in which each element of the domain is related to a single element of the codomain. An important case in this context is the identity, 1_X, on a set X:

$$1_X(x, y) \Leftrightarrow x = y.$$

Given relations $r : X \looparrowright Y$ and $s : Y \looparrowright Z$, we can consider their composite $s \cdot r$ given by, for all $x \in X$ and $z \in Z$, $s \cdot r(x, z)$ whenever there is $y \in Y$ such that $r(x, y)$ and $s(y, z)$.
This composition is associative, that is,

$$r \cdot (s \cdot t) = (r \cdot s) \cdot t,$$

for all $r : Y \looparrowright Z, s : X \looparrowright Y, t : W \looparrowright X$, and the identity relation acts as neutral element:

$$1_X \cdot r = r \quad \& \quad r \cdot 1_Y = r,$$

for all $r : X \rightarrow\!\!\!\!\!+ Y$. Hence sets and relations form a category denoted by Rel that is an ordered-enriched category since, for $r, s : X \rightarrow\!\!\!\!\!+ Y$,

$$r \leq s \Leftrightarrow r \subseteq s,$$

and composition respects this order:

$$r \leq r' \quad \& \quad s \leq s' \Rightarrow r \cdot s \leq r' \cdot s',$$

for all $r, r' : Y \rightarrow Z$ and all $s, s' : X \rightarrow Y$.
Given a relation $r : X \rightarrow\!\!\!\!\!+ Y$ we can consider its *opposite* or *dual* $r^\circ : Y \rightarrow\!\!\!\!\!+ X$:

$$r^\circ(y, x) \Leftrightarrow r(x, y),$$

for all $y \in Y$ and all $x \in X$, and one has, for all $s : X \rightarrow\!\!\!\!\!+ Y$, $r, r' : Y \rightarrow\!\!\!\!\!+ Z$,

$$(r \cdot s)^\circ = s^\circ \cdot r^\circ, \quad (1_X)^\circ = 1_X, \quad (r^\circ)^\circ = r, \quad r \leq r' \Rightarrow r^\circ \leq r'^\circ.$$

Bearing in mind that the usual composition of functions coincides with the composition of the underlying relations, the map $\mathsf{Set} \to \mathsf{Rel}$, that leaves objects unchanged and views functions as relations, is a faithful functor.

Since Rel has ordered hom-sets, it is possible to talk about adjunctions of relations. A relation $r : X \rightarrow\!\!\!\!\!+ Y$ is *left adjoint* if there is a relation $s : Y \rightarrow\!\!\!\!\!+ X$ such that

$$1_X \leq s \cdot r \text{ and } r \cdot s \leq 1_Y.$$

The relation s is called *right adjoint* and the adjunction is denoted by $r \dashv s$. Note that the right adjoint (respectively, left adjoint) is unique since, if also $r \dashv s'$, then $s' = 1_X \cdot s' \leq s \cdot r \cdot s' \leq s \cdot 1_Y = s$ and $s = 1_X \cdot s \leq s' \cdot r \cdot s \leq s' \cdot 1_Y = s'$; consequently $s = s'$.
Another interesting result is that for any map f, f is left adjoint to f°. In fact, maps are precisely the relations that are left adjoints.

Theorem 1.1. *A relation r is left adjoint if and only if it is a function.*

Proof. The first condition of the definition of left adjoint relation guarantees that any element of the domain is related to, at least, one element of the codomain, while the second one ensures that such an element is unique. The reciprocal is also valid since any map $f : X \to Y$ seen as the relation $f : X \relbar\joinrel\mapsto Y$ such that $f(x, y) \Leftrightarrow y = f(x)$, has a right adjoint f°. $\square$

1.2 Ordered sets

A relation r on a set X, $r : X \relbar\joinrel\mapsto X$, is *reflexive* if $1_X \leq r$, it is *transitive* if $r \cdot r \leq r$, it is *antisymmetric* if $r \cdot r^\circ \leq 1_X$ and it is *symmetric* if $r = r^\circ$.
An *ordered set* $(X, \leq)$ is a set X equipped with a reflexive and transitive relation $\leq : X \relbar\joinrel\mapsto X$. Usually this kind of structure is called *pre-ordered set* and adding anti-symmetry we obtain a *partial ordered set* (or *poset*) or a *separated ordered set.*
The dual of any ordered set $X = (X, \leq)$ is the structure $X^{\mathrm{op}} = (X, \leq^\circ)$ obtained just by considering the opposite order relation on the same set. Note also that an ordered set X is a category whose objects are the elements of X and in which there is, at most, one morphism between any two objects.
Any order relation can be described as a function that, to each $x \in X$, associates the set of elements x' of X such that $x \leq x'$, that we will call the *up-set* of x in X and denote by

$$\uparrow\! x = \{x' \in X : x \leq x'\}.$$

Using the same notation, $\mathrm{P}_\uparrow X$ represents the set of all up-sets of X, $\uparrow_X A = \bigcup_{x \in A} \uparrow\! x$ is the *up-closure* of $A \subseteq X$ in X and $A \subseteq X$ is *up-closed* in X if $\uparrow_X A = A$. The dual notions of *down-set* of x in X

$$\downarrow\! x = \{x' \in X : x' \leq x\},$$

of $\mathrm{P}_\downarrow X$, of *down-closure* of a subset of X and of *down-closed* subset of X will also be relevant.
Any set X can be seen as the ordered set $(X, =)$, the sets PX and $\mathrm{P}_\downarrow X$ are ordered by inclusion and $\mathrm{P}_\uparrow X$ is ordered by reverse inclusion.

Examples 1.2. The main examples are the set of true values $\mathsf{2} = \{\mathsf{true}, \mathsf{false}\}$ ordered by $\Rightarrow$, the interval $[0, 1]$ and the extended real half line $P^+ = [0, +\infty]$ with the usual "less or equal" relation and again $P^+ = [0, +\infty]$ but ordered by $\geq$, that is, $[0, +\infty]^{\mathrm{op}}$.

An application $f : X \to Y$ between the ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$ is *order-preserving* or *monotone* if $f \cdot \leq_X \subseteq \leq_Y \cdot f$, that is, for all $x, x' \in X$,

$$x \leq_X x' \Rightarrow f(x) \leq_Y f(x').$$

Given a monotone map $f : X \to Y$, the map $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$, with $f^{\mathrm{op}}(x) = f(x)$ is also monotone.

Example 1.3. The maps $f_{n,\epsilon} : [0, +\infty] \to [0, 1]$, with $n \in [0, +\infty]$ and $\epsilon \in [0, 1]$, defined by:

$$f_{n,\epsilon}(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq n \\ \epsilon & \text{if } n < x \end{cases}$$

are monotone.

Considering the ordered sets as objects and monotone maps as morphisms, we obtain the category Ord, that is also an ordered-enriched category since, for any ordered sets X and Y,

$$\mathsf{Ord}(X, Y) = \{f : X \to Y, f \text{ monotone }\}$$

is ordered pointwise. Hence, a monotone map $f : X \to Y$ is *left adjoint* if there is a monotone map $g : Y \to X$ such that $1_X \leq g \cdot f$ and $f \cdot g \leq 1_Y$, that is, for all $x \in X$ and $y \in Y$,

$$x \leq g \cdot f(x) \text{ and } f \cdot g(y) \leq y.$$

Note that:

1. for any adjunction $f \dashv g$ of monotone maps, one has $f \simeq f \cdot g \cdot f$ and $g \simeq g \cdot f \cdot g$.
2. the monotone map $f : X \to Y$ is left adjoint to $g : Y \to X$ if and only if $f(x) \leq y \Leftrightarrow x \leq g(y)$, for all $x \in X$ and all $y \in Y$.

To show the first statement let $f \dashv g$; then

$$f = f \cdot 1_X \leq f \cdot g \cdot f \leq 1_Y \cdot f = f$$

and

$$g = 1_X \cdot g \leq g \cdot f \cdot g \leq g \cdot 1_Y = g.$$

To prove the second statement, let $f \dashv g$; if $f(x) \leq y$ then $x \leq g \cdot f(x) \leq g(y)$ and if $x \leq g(y)$ then $f(x) \leq f \cdot g(y) \leq y$. Finally, from $f(x) = f(x)$ and $g(y) = g(y)$ we have $x \leq g \cdot f(x)$ and $f \cdot g(y) \leq y$, respectively.

Any adjunction $f \dashv g : Y \to X$ of monotone maps induces another adjunction $g^{\mathrm{op}} \dashv f^{\mathrm{op}}$, where $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$ (and vice-versa) since:

$$1_X \leq g \cdot f \text{ and } f \cdot g \leq 1_Y \Leftrightarrow g \cdot f \leq^\circ 1_X \text{ and } 1_Y \leq^\circ f \cdot g.$$

1.3 Distributors

Besides the monotone maps between ordered sets there is another class of morphisms that will be extremely important for the subsequent development.
Given ordered sets $X = (X, \leq_X)$ and $Y = (Y, \leq_Y)$, a relation $\varphi : X \rightarrow\!\!\!\!\!\!\mapsto Y$ is called *distributor* if $\varphi \cdot \leq_X \subseteq \varphi$ and $\leq_Y \cdot \varphi \subseteq \varphi$; that is, if, for all $x, x' \in X, y, y' \in Y$,

$$(x \leq x' \;\;\&\;\; \varphi(x', y) \Rightarrow \varphi(x, y)) \;\text{ and }\; (\varphi(x, y') \;\;\&\;\; y' \leq y \Rightarrow \varphi(x, y)).$$

To distinguish a distributor from a relation, the usual notation for a distributor φ is $\varphi : X \multimapdotboth Y$.
For any distributor $\varphi : X \multimapdotboth Y$, we have that $\varphi = \varphi \cdot 1_X \subseteq (\varphi \cdot \leq_X)$ and $\varphi = 1_Y \cdot \varphi \subseteq (\leq_Y \cdot \varphi)$. Thus $\varphi = \varphi \cdot \leq_X$ and $\varphi = \leq_Y \cdot \varphi$. As a consequence the identity distributor on $(X, \leq_X)$ is actually $\leq_X$, and will be represented by $\leq_X = (1_X)_* = (1_X)^*$ to avoid confusion with the identity map. The category Dist is defined considering ordered sets as objects and distributors as morphisms and, since the morphisms are relations, this category has also ordered *hom-sets.*

Proposition 1.4. *A relation $\varphi : X \rightarrow\!\!\!\!\!\!\mapsto Y$ is a distributor if and only if the application $\varphi : X^{\mathrm{op}} \to PY$ is monotone and $\varphi(X^{\mathrm{op}}) = \mathrm{P}_{\uparrow} Y$.*

Proof. Suppose that $\varphi : X \rightarrow\!\!\!\!\!\!\mapsto Y$ is a distributor. If $x \leq x'$ in X and $y \in \varphi(x')$ then, by definition of distributor, also $y \in \varphi(x)$. Hence $\varphi(x') \subseteq \varphi(x)$. If $y \in \varphi(x)$ and $y \leq y'$ in Y then $y' \in \varphi(x)$, which means that, for any $x \in X$, $\varphi(x) \in \mathrm{P}_{\uparrow} Y$.
If φ is a monotone map and:

- $x' \leq^{\circ} x$ in X then $\varphi(x') \subseteq \varphi(x)$. Hence $y \in \varphi(x')$ implies $y \in \varphi(x)$, or, in other words, $\varphi(x', y)$ implies $\varphi(x, y)$.

- $y \leq y'$ in Y then $y \in \varphi(x)$ implies $y' \in \varphi(x)$, since $\varphi(x)$ is an up-set. $\square$

A distributor $X \multimapdotboth Y$ can also be described as a monotone map $X^{\mathrm{op}} \times Y \to 2$; in particular, a distributor $\varphi : 1 \multimapdotboth X$ can be seen as a monotone map $\varphi : X \to 2$ that defines an up-closed subset of X, and a distributor $\psi : X \multimapdotboth 1$ can be seen as a monotone map $\psi : X^{\mathrm{op}} \to 2$ that defines a down-closed subset of X.
A distributor $\varphi : X \multimapdotboth Y$ is *left adjoint* if there is a distributor $\psi : Y \multimapdotboth X$ such that $(1_X)_* \subseteq \psi \cdot \varphi$ and $\varphi \cdot \psi \subseteq (1_Y)_*$. In pointwise notation, $\varphi \dashv \psi$ if, for all $x, x' \in X$ and all $y, y' \in Y$,

$$x \leq_X x' \Rightarrow \exists y \in Y : \varphi(x, y) \text{ and } \psi(y, x'), \qquad \psi(y, x) \text{ and } \varphi(x, y') \Rightarrow y \leq_Y y'.$$

1.4 Functors between Ord and Dist

A monotone map gives rise to two distributors through the functors:

- $(\ )_* : \mathsf{Ord} \longrightarrow \mathsf{Dist}$, that has no action on objects and to each monotone map $f : X \to Y$ associates the distributor $f_* : X \multimap Y$ such that

$$f_*(x,y) = \begin{cases} \mathsf{true} & \text{if } f(x) \leq y \\ \mathsf{false} & \text{else} \end{cases}$$

- $(\ )^* : \mathsf{Ord} \longrightarrow \mathsf{Dist}^{\mathrm{op}}$, that, to each ordered set associates the same and to each monotone map $f : X \to Y$ associates the distributor $f^* : Y \multimap X$ such that

$$f^*(y,x) = \begin{cases} \mathsf{true} & \text{if } y \leq f(x) \\ \mathsf{false} & \text{else} \end{cases}$$

Note that the relations f_* and f^* can be defined for any map (not necessarily monotone). Such relations are distributors if and only if f is monotone. Furthermore, the distributors thus generated are adjoints: $f_* \dashv f^*$.

Recalling that $\varphi : 1 \multimap X$ defines an up-closed subset of X, A, and that $\psi : X \multimap 1$ defines a down-closed subset of X, B, we have that:

$$\begin{aligned} \varphi \dashv \psi \quad &\Leftrightarrow \quad (1_1)_* \leq \psi \cdot \varphi \quad \text{and} \quad \varphi \cdot \psi \leq (1_X)_* \\ &\Leftrightarrow \quad (\exists x \in X : x \in A \cap B) \quad \text{and} \quad (\forall x, x' \in X, x \in A \,\&\, x' \in B \Rightarrow x' \leq x) \end{aligned}$$

Then $\varphi \dashv \psi$ if and only if there is some $x \in X$ such that $A = \uparrow x = x_*$ and $B = \downarrow x = x^*$. This result permits us to easily conclude that a left adjoint distributor $\varphi : X \multimap Y$ is representable by some monotone map: let $x \in X$ and consider the composition $\varphi \cdot x_*$ that is a left adjoint distributor of the form $1 \multimap Y$. Hence $\varphi \cdot x_*$ is representable by some $y \in Y$ and considering $f : X \to Y$, that to each $x \in X$ associates an $y \in Y$ such that $\varphi \cdot x_* = y_*$, we have that $\varphi = f_*$. The fact that f_* is a distributor guarantees that f is monotone. Note that the Axiom of Choice was used for the definition of f.

Each of the functors described in the beginning of this section have right adjoints $D : \mathsf{Dist} \to \mathsf{Ord}$ and $U : \mathsf{Dist}^{\mathrm{op}} \to \mathsf{Ord}$, respectively, such that, for any distributor $\varphi : X \multimap Y$:

$$\begin{aligned} D(\varphi) : \quad \mathsf{Dist}(1,X) \quad &\to \quad \mathsf{Dist}(1,Y) \\ \alpha \quad &\mapsto \quad \varphi \cdot \alpha \end{aligned}$$

$$\begin{array}{rcll} U(\varphi): & \mathsf{Dist}(Y,1) & \to & \mathsf{Dist}(X,1) \\ & \beta & \mapsto & \beta\cdot\varphi \end{array}$$

For any ordered set X, the units of such adjunctions give $\eta_X(x) = \uparrow x$ and $\eta'_X(x) = \downarrow x$, for all $x \in X$, respectively. We will refer to these maps as *Yoneda embeddings*.

1.5 Complete ordered sets

An ordered set X is *complete* if every subset of X has an infimum in X or, equivalently, if every up-closed subset of X has an infimum in X. The dual notion is the one of *cocomplete* ordered set X: X is *cocomplete* if every subset of X has a supremum in X, or, if every down-closed subset of X has a supremum in X.
Note that the existence of all infima guarantees the existence of all suprema and vice-versa; hence X is complete if and only if X is cocomplete. In particular, a complete ordered set X has a bottom element, $\bot = \bigvee \emptyset$, and a top element, $\top = \bigwedge \emptyset$.
Also note that the supremum and the infimum are not necessarily unique if the order relation is not antisymmetric but they are unique up to isomorphism.

The sets $\mathrm{P}_{\uparrow}X$ and $\mathrm{P}_{\downarrow}X$ are complete ordered sets. In $\mathrm{P}_{\downarrow}X$, infima are given by intersection and suprema are given by union. In $\mathrm{P}_{\uparrow}X$, since the order is reversed, infima are given by union and suprema are given by intersection. If X is a complete ordered set we can define the maps $\bigvee$ and $\bigwedge$, using the Axiom of Choice:

$$\begin{array}{rcll} \bigvee: & \mathrm{P}_{\downarrow}X & \to & X \\ & A & \mapsto & \vee A \end{array} \qquad \begin{array}{rcll} \bigwedge: & \mathrm{P}_{\uparrow}X & \to & X \\ & A & \mapsto & \wedge A \end{array}$$

Proposition 1.5. *An ordered set is complete if and only if $\bigvee \dashv \downarrow$ (or if and only if $\uparrow \dashv \bigwedge$).*

Proof. The implication '$\Rightarrow$' is obvious. To prove '$\Leftarrow$' suppose that there is a monotone map $\vee$ that is left adjoint to $\downarrow$ and let $A \subseteq X$. Since $\vee A \leq \vee A$ then $A \subseteq \downarrow \vee A$, that is, $\vee A$ is an upper bound of A. If x is also an upper bound of A we have that $A \subseteq \downarrow x$ and, consequently, $\vee A \leq x$. Finally, note that

$$\begin{aligned} \vee_X \dashv \downarrow_X &\Leftrightarrow \vee_{X^{\mathrm{op}}} \dashv \downarrow_{X^{\mathrm{op}}} \\ &\Leftrightarrow (\downarrow_{X^{\mathrm{op}}})^{\mathrm{op}} \dashv (\vee_{X^{\mathrm{op}}})^{\mathrm{op}} \\ &\Leftrightarrow \uparrow_X \dashv \wedge_X. \end{aligned}$$

$\square$

A monotone map is called a *sup-map* if it preserves suprema and it is called *inf-map* if it preserves infima.

Proposition 1.6. *Let $f : X \to Y$ be a monotone map. If f is left adjoint then it is a sup-map (and if it is right adjoint then it is an inf-map). Moreover, if X is complete then these implications are, in fact, equivalences.*

Proof. Let $f : X \to Y$ be a left adjoint monotone map and $A \subseteq X$ with supremum $\bigvee A$. For all $x \in A$, $x \leq \bigvee A$ then, since f is monotone $f(x) \leq f(\bigvee A)$. If $y \in Y$ is an upper bound of $f(A)$ and g is right adjoint to f, then, for all $x \in A$,

$$\begin{aligned} f(x) \leq y \ &\Leftrightarrow x \leq g(y) \\ &\Leftrightarrow \bigvee A \leq g(y) \\ &\Leftrightarrow f(\bigvee A) \leq y. \end{aligned}$$

To prove the second statement let X be a complete ordered set and suppose that f is a sup-map. A right adjoint to f is given by the composition $\bigvee_X \cdot \mathrm{P}_{\downarrow}(f^{-1}) \cdot \downarrow_Y$,

$$\begin{tikzcd} \mathrm{P}_{\downarrow}X \arrow[d, "\bigvee_X"'] & \mathrm{P}_{\downarrow}Y \arrow[l, "\mathrm{P}_{\downarrow}f^{-1}"'] \\ X \arrow[r, "f"'] & Y \arrow[u, "\downarrow_Y"'] \end{tikzcd}$$

where $\mathrm{P}_{\downarrow}f^{-1}(B) = \downarrow\{x \in X : f(x) \in B\}$, for every down-closed subset B of Y. $\square$

1.6 Completely distributive ordered sets

A *completely distributive* ordered set (see [Ran52, FW90]) is a complete ordered set in which arbitrary joins distribute over arbitrary meets.

Definition 1.7. An ordered set X is *completely distributive* if it is complete and $\bigvee : \mathrm{P}_{\downarrow}X \to X$ preserves all infima.

By Proposition 1.6, in a completely distributive ordered set X the map $\bigvee_X$ is a right adjoint. This means that there is a monotone map $\Downarrow_X$ such that:

$$\begin{aligned} \Downarrow_X \dashv \bigvee_X \quad &\Leftrightarrow \quad 1_X \leq \bigvee_X \cdot \Downarrow_X \quad \text{and} \quad \Downarrow_X \cdot \bigvee_X \leq 1_{\mathrm{P}_{\downarrow}X} \\ &\Leftrightarrow \quad \forall x \in X, \forall A \in \mathrm{P}_{\downarrow}X, \Downarrow_X x \subseteq A \Leftrightarrow x \leq \bigvee_X A. \end{aligned}$$

Definition 1.8. Assume X is a complete ordered set and $x, y \in X$. Then x is *totally below* y, written as $x \ll y$, if and only if, for all $A \in \mathrm{P}_{\downarrow}X$,

$$y \leq \bigvee A \Rightarrow x \in A.$$

We recall some key facts about totally below relations that can be found in [Fla97]:

Lemma 1.9. *Let $\ll$ be a totally below relation defined in a complete ordered set X. Then, for x, y, z in X:*

1. $x \ll y \Rightarrow x \leq y$;
2. $x \leq y \ll z \Rightarrow x \ll z$;
3. $x \ll y \leq z \Rightarrow x \ll z$;
4. $x \ll y \Rightarrow \exists z \in X : x \ll z \ll y$.

If X is completely distributive then, for any $x \in X$, we have that:

$$\Downarrow x = \cap\{A \in \mathrm{P}_{\downarrow}X : x \leq \bigvee A\}.$$

Therefore

$$y \in \Downarrow x \quad \Leftrightarrow \quad \forall A \in \mathrm{P}_{\downarrow}X, x \leq \bigvee A \Rightarrow y \in A \quad \Leftrightarrow \quad y \ll x.$$

Theorem 1.10. *A complete ordered set X is completely distributive if and only if $x = \bigvee\{y \in X : y \ll x\}$, for every $x \in X$.*

Proof. Let X be a completely distributive ordered set, $x \in X$ and $A = \{y \in X : y \ll x\}$. From (1) of Lemma 1.9 we conclude that x is an upper bound of A. Since $\Downarrow \dashv \bigvee$, $x \leq \bigvee A \Leftrightarrow \Downarrow x \subseteq A$; hence, if $y \in \Downarrow x$ then $y \ll x$, that is $y \in A$.
To prove the reciprocal implication, suppose that any element x of X is the supremum of the set $\Downarrow x = \{y \in X : y \ll x\}$. The application $\Downarrow$ is monotone due to (3) of Lemma 1.9. The inequality $1 \leq \bigvee \cdot \Downarrow$ is obvious and, for any $S \in \mathrm{P}_{\downarrow}X$,

$$\Downarrow \cdot \bigvee S = \{y \in X : y \ll \bigvee S\} \subseteq S. \qquad \square$$

Note that if X is a completely distributive ordered set then X^{op} is also a completely distributive ordered set.

Examples 1.11. The complete ordered set $\mathsf{2}$ is completely distributive considering $x \ll y \Leftrightarrow y = \mathsf{true}$. The complete ordered sets $[0, +\infty]$ and $[0, 1]$ are completely distributive for the totally below relation $x \ll y$ if and only if $x < y$. In $[0,+\infty]^{\text{op}}$, $x \ll y$ if and only if $x > y$.

1.7 The complete distributive ordered set Δ

In this section we consider the set

$$\Delta = \{f : [0,+\infty] \to [0,1] : f(x) = \bigvee_{y<x} f(y)\}$$

of *distribution functions* and show that it is a completely distributive ordered set. The condition that characterizes the functions in Δ means that those functions are left continuous. Note that every left continuous function is, in particular, monotone.

In the sequel we will consider the complete ordered sets $[0,+\infty] = ([0,+\infty], \leq^\circ)$ and $[0,1] = ([0,1], \leq)$, that are both completely distributive. For the relation $\leq$ given by $f \leq g$ if, for all $x \in [0,+\infty]$, $f(x) \leq g(x)$, Δ is a separated ordered set. This is indeed a reflexive and transitive relation but it is also antisymmetric.

Let $A = \{h_i \in \Delta : i \in I\} \subseteq \Delta$; the supremum of A is given by h, where, for all $x \in X$,

$$h(x) = \bigvee_{i \in I} h_i(x),$$

which, indeed, belongs to Δ. In fact, since $[0,1]$ is a complete ordered set then $\bigvee_{i\in I} h_i(x)$ exists and it is left continuous:

$$\begin{aligned} h(x) =& \bigvee_{i\in I} h_i(x) \\ =& \bigvee_{i\in I}\bigvee_{y<x} h_i(y) \\ =& \bigvee_{y<x}\bigvee_{i\in I} h_i(y) \\ =& \bigvee_{y<x} h(y) \end{aligned}$$

for all $x \in X$.

Hence $(\Delta, \leq)$ is cocomplete. The infimum of any subset of Δ can not, in general, be obtained pointwise as the supremum, as can be seen by considering the distribution functions of the form

$$f_{n,1}(x) = \begin{cases} 0 & \text{if } 0 \le x \le n \\ 1 & \text{if } n < x \end{cases}$$

The pointwise infimum, h, of this set of functions is:

$$h(x) = \begin{cases} 0 & \text{if } x < +\infty \\ 1 & \text{if } x = +\infty \end{cases}$$

that is not in Δ because it is not left continuous.
It remains to prove the complete distributivity of Δ. The maps $f_{n,\epsilon}$, with $0 \le \epsilon \le 1$ and $0 \le n$, introduced in Example 1.3, are in Δ and they allow a more simplified description of Δ and its properties.
The totally below relation in Δ (see [Fla97]) gives

$$\forall f, f_{n,\epsilon} \in \Delta, \quad f_{n,\epsilon} \ll f \Leftrightarrow \epsilon < f(n),$$

and any element f of Δ can be described as the supremum of those $f_{n,\epsilon}$ such that $f_{n,\epsilon} \ll f$. In fact, for any $x \in [0, +\infty]$,

$$\begin{aligned} f(x) &= \bigvee_{y<x} f(y) \\ &= \bigvee_{\substack{y<x \\ \alpha>0}} f(y) - \alpha \\ &= \bigvee_{\substack{y<x \\ \alpha>0}} f_{y,f(y)-\alpha}(x) \end{aligned}$$

and, obviously, $f_{y,f(y)-\alpha} \ll f$.Then, by Theorem 1.10, Δ is completely distributive.

1.8 Quantales

Definition 1.12. A *quantale* $(\mathsf{V}, \le, \otimes, k)$ consists of a separated and complete ordered set $(\mathsf{V}, \le)$ and a binary operation $\otimes$ defined in V such that $(\mathsf{V}, \otimes, k)$ is a commutative monoid with identity k and, for any $u \in \mathsf{V}$, the map $u \otimes - : \mathsf{V} \to \mathsf{V}$ preserves suprema, that is for any $u, u_i \in \mathsf{V}$, with $i \in I$,

$$u \otimes (\bigvee_{i \in I} u_i) = \bigvee_{i \in I} (u \otimes u_i).$$

Since, for each $u \in \mathsf{V}$, the map $u \otimes - : \mathsf{V} \to \mathsf{V}$ preserves suprema, it has a right adjoint denoted by $\mathsf{hom}(u,-) : \mathsf{V} \to \mathsf{V}$. Consequently, for all $u, v, w \in \mathsf{V}$,

$$u \otimes v \leq w \;\Leftrightarrow\; v \leq \mathsf{hom}(u, w).$$

This adjoint is given by

$$\mathsf{hom}(u, v) = \bigvee\{w \in \mathsf{V} : u \otimes w \leq v\},$$

for any $u, v \in \mathsf{V}$.

Examples 1.13.

1. The two-element Boolean algebra 2 is a quantale with $\otimes = \wedge$ and $k = \mathsf{true}$.
2. In general, every frame is a quantale with $\otimes = \wedge$ and $k = \top$. In particular PX is a quantale.
3. Another example of 2 is the interval $[0, 1]$ which is a quantale for $\otimes = \wedge$ and $k = 1$.
4. The ordered set $[0, 1]$ is also a quantale when considering the usual product . The right adjoint is given by a "division" $\mathsf{hom}(x, y) = y \oslash x$, where $y \oslash 0 = 1$ and $y \oslash x = \min\{y/x, 1\}$ if $x \neq 0$.
5. The real half-line $[0, +\infty]$ ordered by the "great or equal" relation, $\geq$, is a quantale with tensor given by the usual addition such that $x + \infty = \infty + x = \infty$, for all $x, y \in [0, +\infty]$. Obviously, $k = 0$ is the neutral element. The right adjoint in this case is $\mathsf{hom}(x, y) = y \ominus x$ where $y \ominus x = \max\{y - x, 0\}$. Since the order is the opposite to the usual one, 0 is the top element and $+\infty$ is the bottom element, and we have $\bigvee = \inf$ and $\bigwedge = \sup$.
6. If $(M, \cdot, e)$ is a commutative monoid, the complete ordered set PM is a quantale for the product given by $A \otimes B = \{a \cdot b : a \in A, b \in B\}$.

Definition 1.14. Let $(\mathsf{V}, \leq, \otimes, k)$ and $(\mathsf{W}, \leq, \oplus, l)$ be quantales. A monotone map $F : \mathsf{V} \to \mathsf{W}$ is a *morphism of quantales* if F preserves $\otimes$, $\bigvee$ and the identity, that is, for all $u, v, v_i \in \mathsf{V}$ with $i \in I$:

1. $F(u \otimes v) = F(u) \oplus F(v)$;
2. $F(\bigvee\limits_{i\in I} v_i) = \bigvee\limits_{i\in I} F(v_i)$;
3. $F(k) = l$.

It turns out that for many applications it is enough to have inequalities above; in this case, we say that F is a *lax morphism of quantales*. That is, a lax morphism of quantales $F : \mathsf{V} \to \mathsf{W}$ only needs to satisfy, for all $u, v \in \mathsf{V}$,

$$F(u) \oplus F(v) \leq F(u \otimes v), \qquad\qquad l \leq F(k).$$

Note that the inequality $\bigvee_{i\in I} F(v_i) \leq F(\bigvee_{i\in I} v_i)$ is equivalent to the monotonicity of F.

Examples 1.15. 1. The map $I : \mathsf{2} \to [0, +\infty]$, interpreting false as $+\infty$ and true as 0, is a morphism of quantales, and note that I is indeed monotone since we consider the "greater or equal" relation $\geq$ on $[0, +\infty]$. Furthermore, $I : \mathsf{2} \to [0, +\infty]$ has a left and a right adjoint given by

$$O : [0, +\infty] \to \mathsf{2}, \quad x \mapsto \begin{cases} \mathsf{true} & \text{if } x < +\infty \\ \mathsf{false} & \text{if } x = +\infty \end{cases} \qquad\qquad P : [0, +\infty] \to \mathsf{2} \quad x \mapsto \begin{cases} \mathsf{true} & \text{if } x = 0 \\ \mathsf{false} & \text{if } x > 0 \end{cases},$$

respectively. Here $O : [0, +\infty] \to \mathsf{2}$ is a morphism of quantales as well, but $P : [0, +\infty] \to \mathsf{2}$ is only a lax morphism of quantales. This construction can be generalised to an arbitrary quantale V. Firstly, the map $I : \mathsf{2} \to \mathsf{V}$ interpreting false as $\bot$ and true as k is a morphism of quantales. Since I preserves suprema it has a right adjoint $P : \mathsf{V} \to \mathsf{2}$. I preserves the top element precisely if $k = \top$, and in this case I preserves all infima and, therefore, has a left adjoint O. Furthermore, the left and the right adjoint to I are given by

$$O : \mathsf{V} \to \mathsf{2} \quad x \mapsto \begin{cases} \mathsf{true} & \text{if } x \neq \bot \\ \mathsf{false} & \text{if } x = \bot \end{cases} \qquad\qquad P : \mathsf{V} \to \mathsf{2} \quad x \mapsto \begin{cases} \mathsf{true} & \text{if } x \geq k \\ \mathsf{false} & \text{else} \end{cases}$$

respectively. Being left adjoint, $O : \mathsf{V} \to \mathsf{2}$ preserves suprema and one easily verifies that $O(k) = \mathsf{true}$; but O is not, in general, a lax morphism of quantales since one only has $O(u \otimes v) \leq O(u) \wedge O(v)$ (the other inequality is obtained whenever $u \otimes v = \bot \Rightarrow u = \bot$ or $v = \bot$, for all $u, v \in \mathsf{V}$). Finally, the right adjoint $P : \mathsf{V} \to \mathsf{2}$ is a lax morphism of quantales.

2. The bijection

$$\begin{aligned} E : [0, +\infty] &\to [0, 1] \\ x &\mapsto \exp(-x), \end{aligned}$$

where $\exp(-\infty) = 0$, is a morphism of quantales, and so is its inverse

$$\begin{aligned} L : [0, 1] &\to [0, +\infty] \\ x &\mapsto -\ln(x), \end{aligned}$$

where $-\ln(0) = +\infty$.

1.9 The quantale Δ

We have already seen that Δ is a completely distributive ordered set. Consider now the binary operation $\otimes$ defined in Δ as follows:

$$f \otimes g(x) = \bigvee_{y+z \leq x} (f(y) * g(z))$$

for all $f, g \in \Delta$ and all $x \in [0, +\infty]$. The operation $*$ represents a commutative and associative product in $[0, 1]$, with neutral element κ, such that $\lambda * - : [0, 1] \to [0, 1]$ preserves suprema, for all $\lambda \in [0, 1]$. In the sequel we will often consider the usual product in $[0, 1]$, that will be denoted by $\cdot$ to highlight the difference. In order to prove that $f \otimes g$ is in Δ it is necessary to check that $f \otimes g(x) = \bigvee_{y<x} f \otimes g(y)$, for any $x \in [0, +\infty]$.

$$\begin{aligned} f \otimes g(x) &= \bigvee_{q+s \leq x} (f(q) * g(s)) \\ &= \bigvee_{q+s \leq x} [\bigvee_{u<q} (f(u) * \bigvee_{v<s} g(v))] \\ &= \bigvee_{q+s \leq x} \bigvee_{u<q} \bigvee_{v<s} [(f(u) * g(v))] \\ &= \bigvee_{u+v<x} [(f(u) * g(v))] \\ &= \bigvee_{y<x} \bigvee_{u+v \leq y} [(f(u) * g(v))] \\ &= \bigvee_{y<x} f \otimes g(y). \end{aligned}$$

Furthermore, $\otimes$ is commutative and associative in Δ and

$$k(x) = \begin{cases} \kappa & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is the identity in Δ. Thus $(\Delta, \otimes)$ is a commutative monoid. Moreover, for any $f \in \Delta$, any family $g_i, i \in I$, of elements of Δ, and any $x \in [0, +\infty]$,

$$\begin{aligned} (f \otimes \bigvee_{i\in I} g_i)(x) &= \bigvee_{y+z\leq x} (f(y) * \bigvee_{i\in I} g_i(z)) \\ &= \bigvee_{i\in I} \bigvee_{y+z\leq x} (f(y) * g_i(z)) \\ &= \bigvee_{i\in I} (f \otimes g_i)(x) \end{aligned}$$

which means that $\otimes$ preserves suprema and, consequently, that $(\Delta, \leq, \otimes, k)$ is a quantale. Note that $f_{n,\epsilon} \otimes f_{n',\epsilon'} = f_{n+n',\epsilon*\epsilon'}$.

Since, for any $f \in \Delta$, $f \otimes -$ preserves suprema, it has a right adjoint $\mathsf{hom}(f, -)$. For a generic element g in Δ,

$$\begin{aligned} \mathsf{hom}(f, g) &= \bigvee\{h \in \Delta : f \otimes h \leq g\} \\ &= \bigvee\{h \in \Delta : \forall x \in [0, +\infty], \bigvee_{y+z\leq x} f(y) * h(z) \leq g(x)\} \\ &= \bigvee\{h \in \Delta : \forall x \in [0, +\infty], (y + z \leq x \Rightarrow f(y) * h(z) \leq g(x))\}. \end{aligned}$$

If $f = f_{n,\epsilon}$ then, for any $g \in \Delta$,

$$\begin{aligned} \mathsf{hom}(f_{n,\epsilon}, g) &= \bigvee\{h \in \Delta : \forall x \in [0, +\infty], (y + z \leq x \Rightarrow f_{n,\epsilon}(y) * h(z) \leq g(x))\} \\ &= \bigvee\{h \in \Delta : \forall x \geq 0, (n + z \leq x \Rightarrow \epsilon * h(z) \leq g(x))\} \\ &= \bigvee\{h \in \Delta : \forall x \geq 0, \bigvee_{z\leq x\ominus n} h(z) \leq g(x) \oslash \epsilon\} \\ &= \bigvee\{h \in \Delta : \forall x \geq 0, h(x \ominus n) \leq g(x) \oslash \epsilon\}. \end{aligned}$$

Note that $\mathsf{hom}(f_{n,\epsilon}, g) = k$ when $g \geq f_{n,\epsilon}$.

If, moreover, $g = f_{n',\epsilon'}$, then

$$\mathsf{hom}(f_{n,\epsilon}, f_{n',\epsilon'}) = f_{n'\ominus n,\epsilon'\oslash\epsilon}.$$

Finally, since we can consider any element f of Δ as the supremum of those $f_{n,\epsilon} \ll f$

in Δ,

$$\begin{aligned}\mathsf{hom}(f,g) &= \mathsf{hom}(\bigvee_{f_{n,\epsilon}\ll f} f_{n,\epsilon}, g)\\ &= \bigwedge_{f_{n,\epsilon}\ll f} \mathsf{hom}(f_{n,\epsilon}, g)\end{aligned}$$

for any $g \in \Delta$.

We call $f \in \Delta$ *finite* if $f(+\infty) = 1$. Certainly, if f is finite, then so is every $g \in \Delta$ with $f \leq g$; and one also easily verifies that $f \otimes g$ is finite if both $f, g \in \Delta$ are so.

In the sequel we will consider that $*$ is the usual product in $[0,1]$.

There are some interesting morphisms of quantales involving Δ.

Examples 1.16. The quantale $[0,+\infty]$ embeds canonically into Δ via

$$\begin{aligned}I_\infty : [0,+\infty] &\to \Delta,\\ n &\mapsto f_{n,1}.\end{aligned}$$

Moreover, for all $n, m \in [0,+\infty]$,

$$f_{0,1} = k, \qquad\qquad f_{n+m,1} = f_{n,1} \otimes f_{m,1},$$

and I_∞ preserves suprema since it has a right adjoint

$$\begin{aligned}P_\infty : \Delta &\to [0,+\infty]\\ f &\mapsto \inf\{n \in [0,+\infty] : f(n) = 1\}.\end{aligned}$$

Consequently, I_∞ is a morphism of quantales. The right adjoint P_∞ satisfies

$$P_\infty(k) = 0, \qquad\qquad P_\infty(f \otimes g) = P_\infty(f) + P_\infty(g),$$

for all $f, g \in \Delta$; however P_∞ does not preserve suprema and, therefore, it is only a lax morphism of quantales. Furthermore, I_∞ has also a left adjoint

$$\begin{aligned}O_\infty : \Delta &\to [0,+\infty]\\ f &\mapsto \sup\{n \in [0,+\infty] : f(n) = 0\},\end{aligned}$$

which, being left adjoint, preserves suprema and also satisfies

$$O_\infty(k) = 0, \qquad\qquad O_\infty(f \otimes g) = O_\infty(f) + O_\infty(g),$$

for all $f, g \in \Delta$. Therefore O_∞ is a morphism of quantales.

$$\begin{tikzcd}
{[0,+\infty]} \arrow[rr, "I_\infty" description] & & \Delta \arrow[ll, bend right, "O_\infty"'] \arrow[ll, bend left, "P_\infty"]
\end{tikzcd}$$

Chapter 2

Quantale enriched categories

This chapter recalls a large scope of concepts and results of the theory of categories enriched in a quantale. Probabilistic metric spaces are characterised as enriched categories over the quantale of distribution functions. On this subject we refer to the book *Basic concepts of enriched categories* of G. K. Kelly, but also the introductory sections of [Tho08], [CH08] and [HT10], among others.

2.1 V-Relations

Let $(\mathsf{V}, \leq, \otimes)$ be a quantale. Consider two sets X and Y. We say that r is a V-*relation* from X to Y, written as $r : X \relbar\!\!\!\mapsto Y$, if r is a function from $X \times Y$ to V.
Given two V-relations $r : X \relbar\!\!\!\mapsto Y$ and $s : Y \relbar\!\!\!\mapsto Z$, the composition $s \cdot r : X \relbar\!\!\!\mapsto Z$ is a V-relation defined by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z),$$

for all $x \in X$ and $z \in Z$. Since the composition is associative and the identity morphism in X is the V-relation

$$1_X(x, y) = \begin{cases} k & \text{if } x = y \\ \perp & \text{if } x \neq y \end{cases}$$

we have a category denoted by $\mathsf{V}\text{-}\mathsf{Rel}$.
The order in V induces an order on the set of morphisms from X to Y. If r and s are two V-relations from X to Y, we have

$$r \leq s \Leftrightarrow r(x, y) \leq s(x, y),$$

for all $x \in X$ and $y \in Y$. Given a V-relation $r : X \relbar\!\!\!\mapsto Y$ we define $r^\circ : Y \relbar\!\!\!\mapsto X$ by

$r^\circ(y,x) = r(x,y)$, for all $x \in X$ and $y \in Y$; therefore,

$$1_X^\circ = 1_X, \qquad (s \cdot r)^\circ = r^\circ \cdot s^\circ, \qquad (r^\circ)^\circ = r,$$

and

$$r \leq s \quad \Leftrightarrow \quad r^\circ \leq s^\circ,$$

which means that the functor $(-)^{\mathrm{op}} : \mathsf{V}\text{-}\mathsf{Rel}(X,Y) \to \mathsf{V}\text{-}\mathsf{Rel}(Y,X)$ preserves the order. Since the product $\otimes$ in V preserves suprema, for any V-relation $r : X \rightarrowtriangle\!\!\!\!\!\!\!\!\;\;\; Y$, also the maps $(-) \cdot r : \mathsf{V}\text{-}\mathsf{Rel}(Y,Z) \to \mathsf{V}\text{-}\mathsf{Rel}(X,Z)$ and $r \cdot (-) : \mathsf{V}\text{-}\mathsf{Rel}(Z,X) \to \mathsf{V}\text{-}\mathsf{Rel}(Z,Y)$ preserve suprema. Hence, both have right adjoints, respectively, *extension* and *lifting*:

$$\begin{tikzcd} X \arrow[r, "t", "|"{marking}] \arrow[d, "r"', "|"{marking}] & Z \\ Y \arrow[ur, dotted, "|"{marking}, "t \,\bullet\!\!-\, r"'] & \end{tikzcd} \qquad\qquad \begin{tikzcd} Y & Z \arrow[l, "t"', "|"{marking}] \arrow[dl, dotted, "|"{marking}, "r \,-\!\!\bullet\, t"] \\ X \arrow[u, "r", "|"{marking}] & \end{tikzcd}$$

Any function $f : X \to Y$ is a V-relation when viewed as follows

$$f(x,y) = \begin{cases} k & \text{if } y = f(x) \\ \perp & \text{if } y \neq f(x). \end{cases}$$

In other words, there is an embedding $\mathsf{Set} \to \mathsf{V}\text{-}\mathsf{Rel}$.
The composition of relations is simpler when one of the relations is a function:

$$\begin{aligned} f \cdot r(x,z) &= \bigvee_{y \in Y} r(x,y) \otimes f(y,z) & \qquad s \cdot f(y,w) &= \bigvee_{z \in Z} f(y,z) \otimes s(z,w) \\ &= \bigvee_{y \in f^{-1}(z)} r(x,y) & &= s(f(y),w) \end{aligned}$$

for $r : X \rightarrowtriangle\!\!\!\!\!\!\!\;\;\; Y$, $f : Y \to Z$, $s : Z \rightarrowtriangle\!\!\!\!\!\!\!\;\;\; W$, $x \in X$, $z \in Z$, $y \in Y$ and $w \in W$.
Moreover, for each application $f : X \to Y$, we have $1_X \leq f^\circ \cdot f$ and $f \cdot f^\circ \leq 1_Y$, that is, $f \dashv f^\circ$ in $\mathsf{V}\text{-}\mathsf{Rel}$. Note that for any V-relations $r : X \rightarrowtriangle\!\!\!\!\!\!\!\;\;\; Y$ and $s : Y \rightarrowtriangle\!\!\!\!\!\!\!\;\;\; X$ we have $r \dashv s$ if and only if

$$1_X \leq s \cdot r \quad \text{and} \quad r \cdot s \leq 1_Y.$$

In pointwise notation, r is left adjoint to s if, for all $x \in X$,

$$k \leq \bigvee_{y \in Y} r(x,y) \otimes s(y,x) \qquad \text{and} \qquad \forall y, y' \in Y, (y \neq y' \Rightarrow s(y,x) \otimes r(x,y') = \perp).$$

Lemma 2.1. *Let $r, r' : X \rightarrowtriangle\!\!\!\!\!\!\!\;\;\; Y$ and $s, s' : Y \rightarrowtriangle\!\!\!\!\!\!\!\;\;\; X$ be V-relations such that $r \dashv s$ and $r' \dashv s'$. Then $r \leq r'$ if and only if $s' \leq s$. Therefore, if $r \leq r'$ and $s \leq s'$ then $r = r'$*

and $s = s'$.

The proof of this statement can be adapted from the proof of Lemma 2.17, that is a more general result.
The following theorem is due to [CH08].

Theorem 2.2. *In a quantale* V *with* $k = \top$, *each left adjoint* V*-relation is a map if and only if* k *and* $\perp$ *are the only elements of* V *whose tensor product is* $\perp$ *and the supremum is* k *and, for any* $u, v \in \mathsf{V}$, $u \otimes v = k$ *implies* $u = v = k$.

Examples 2.3. The category $\mathsf{2}$-Rel is just Rel. The category $[0,+\infty]$-Rel is the category of generalised distances also called numeric or fuzzy relations.

Every morphism of quantales $F : \mathsf{V} \to \mathsf{W}$ induces a functor $F : \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{W}\text{-}\mathsf{Rel}$ that transforms a V-relation $r : X \rightarrow\!\!\!\!\!\!\mapsto Y$ in the W-relation obtained by composing $F \cdot r$. This functor preserves the order between V-relations, that is, F is a 2-functor. The composition with the embedding $\mathsf{Set} \to \mathsf{V}\text{-}\mathsf{Rel}$ gives us the commutative diagram:

$$\begin{tikzcd} \mathsf{Set} \arrow[r] \arrow[dr] & \mathsf{V}\text{-}\mathsf{Rel} \arrow[d, "F"] \\ & \mathsf{W}\text{-}\mathsf{Rel} \end{tikzcd}$$

If the map $F : \mathsf{V} \to \mathsf{W}$ is only a lax morphism of quantales, also $F : \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{W}\text{-}\mathsf{Rel}$ is a lax functor, that is, $Ff \cdot Fg \leq F(f \cdot g)$ and $1_{FX} \leq F(1_X)$. In both cases we have that $F(r^\circ) = (Fr)^\circ$.

Examples 2.4.

1. The morphism of quantales $I : \mathsf{2} \to [0,+\infty]$ introduced in Examples 1.15 (1) induces a functor $I : \mathsf{Rel} \to [0,+\infty]$-$\mathsf{Rel}$ which transforms a relation $r : X \rightarrow\!\!\!\!\!\!\mapsto Y$ in the $[0,+\infty]$-relation such that

$$r(x,y) = \begin{cases} 0 & \text{if } r(x,y) \\ +\infty & \text{else.} \end{cases}$$

   We have seen that this morphism has a left and a right adjoint, named O and P, respectively. To each $[0,+\infty]$-relation r, the left adjoint O assigns the relation

$$r(x,y) = \begin{cases} \mathsf{false} & \text{if } r(x,y) = +\infty \\ \mathsf{true} & \text{else} \end{cases}$$

and the right adjoint P, assigns the relation

$$r(x,y) = \begin{cases} \mathsf{true} & \text{if } r(x,y) = 0 \\ \mathsf{false} & \text{else} \end{cases}.$$

The left adjoint O is a functor but the right adjoint P is just a lax functor.

2. In 1.15 (1) we have also seen that, for any quantale V, the morphism of quantales $I : \mathsf{2} \to \mathsf{V}$ that takes false to $\perp$ and true to k, induces the functor $I : \mathsf{Rel} \to \mathsf{V}\text{-}\mathsf{Rel}$ and that it has a left and a right adjoint, also named O and P, respectively. The right adjoint P is a lax morphism of quantales so it induces the lax functor $P : \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{Rel}$ which interprets a V-relation $r : X \rightarrow\!\!\!\!\!+ Y$ as the relation

$$r(x,y) = \begin{cases} \mathsf{true} & \text{if } r(x,y) \geq k \\ \mathsf{false} & \text{else} \end{cases}$$

for any $x \in X$ and $y \in Y$.

The left adjoint is lax morphism of quantales if and only if $u \otimes v = \perp \Rightarrow u = \perp$ or $v = \perp$. In these conditions, it induces the lax functor $O : \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{Rel}$ which interprets a V-relation $r : X \rightarrow\!\!\!\!\!+ Y$ as the relation

$$r(x,y) = \begin{cases} \mathsf{false} & \text{if } r(x,y) = \perp \\ \mathsf{true} & \text{else} \end{cases}$$

for any $x \in X$ and $y \in Y$.

2.2 V-Categories

A V-*category* is a structure (X, a), where X is a set and $a : X \rightarrow\!\!\!\!\!+ X$ is a reflexive and transitive V-relation, that is $1_X \leq a$ and $a \cdot a \leq a$; or, using pointwise notation, for any $x, y, z \in X$:

$$k \leq a(x,x) \qquad \text{and} \qquad a(x,y) \otimes a(y,z) \leq a(x,z).$$

We actually have that $a \cdot a = a$ because $1_X \leq a$ implies $a \leq a \cdot a$.

A V-*functor* $f : (X, a) \to (Y, b)$ is an application between two V-categories such that

$f \cdot a \leq b \cdot f$, or, in pointwise notation, for all $x, x' \in X$,

$$a(x, x') \leq b(f(x), f(x')).$$

Considering the usual function composition and, for each X, the identity function on X, we have the category $\mathsf{V}\text{-}\mathsf{Cat}$. One can easily verify that $(\mathsf{V}, \mathsf{hom})$ is an object of $\mathsf{V}\text{-}\mathsf{Cat}$, whereas hom is a reflexive and transitive relation in V. In fact, for all $u \in \mathsf{V}$, from $u \otimes k = u$ we get $k \leq \mathsf{hom}(u, u)$; on the other hand, as, for all $u, v \in \mathsf{V}$, $u \otimes \mathsf{hom}(u, v) \leq v$, we have

$$u \otimes \mathsf{hom}(u, v) \otimes \mathsf{hom}(v, w) \leq v \otimes \mathsf{hom}(v, w) \leq w$$

and, finally,

$$\mathsf{hom}(u, v) \otimes \mathsf{hom}(v, w) \leq \mathsf{hom}(u, w),$$

for all $u, v, w \in \mathsf{V}$.
To every V-category $X = (X, a)$ one associates its *dual* $X^{\mathrm{op}} = (X, a^\circ)$ where $a^\circ(x, y) = a(y, x)$, for all $x, y \in X$. Since a V-functor $f : (X, a) \to (Y, b)$ can also be seen as a V-functor of type $(X, a^\circ) \to (Y, b^\circ)$, we actually obtain a functor $(-)^{\mathrm{op}} : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$.
There is a canonical forgetful functor $\mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{Ord}$ which sends a V-category $X = (X, a)$ to the ordered set $(X, \leq)$ where, for $x, x' \in X$,

$$x \leq x' \Leftrightarrow k \leq a(x, x').$$

A V-category $X = (X, a)$ is called *symmetric* whenever $X = X^{\mathrm{op}}$, which amounts to saying that $a(x, y) = a(y, x)$ for all $x, y \in X$. A V-category $X = (X, a)$ is called *separated* if the underlying order is antisymmetric, that is, if $x \simeq y$ implies $x = y$, for all $x, y \in X$.
Let $f, g : (X, a) \to (Y, b)$ and consider $f \leq g$ whenever, for all $x \in X$, $f(x) \leq g(x)$. Then $\mathsf{V}\text{-}\mathsf{Cat}$ has ordered hom-sets and composition from either side preserves this order. One fundamental consequence is the possibility to talk about adjunctions. Here a pair of V-functors $f : (X, a) \to (Y, b)$ and $g : (Y, b) \to (X, a)$ forms an adjunction, $f \dashv g$, whenever $1_X \leq g \cdot f$ and $f \cdot g \leq 1_Y$. Equivalently, $f \dashv g$ if and only if, for all $x \in X$ and $y \in Y$,

$$b(f(x), y) = a(x, g(y)),$$

and the formula above explains why one calls f left adjoint and g right adjoint. We also recall that a left adjoint f has at most one right adjoint since $f \dashv g$ and $f \dashv g'$ imply $g \simeq g'$; and dually, $f \dashv g$ and $f' \dashv g$ imply $f \simeq f'$.
The canonical forgetful functor $\mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$, $(X, a) \mapsto X$ is topological (see [AHS90]) where the initial structure on X with respect to the family $f_i : X \to (X_i, a_i)$, $i \in I$, is

given by

$$a(x,x') = \bigwedge_{i\in I} a_i(f_i(x), f_i(x')),$$

for all $x, x' \in X$. Hence, $\mathsf{V}\text{-}\mathsf{Cat}$ admits all limits and all colimits which are, moreover, preserved by $\mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$. In particular, the product $X \times Y$ of V-categories $X = (X, a)$ and $Y = (Y, b)$ can be constructed by taking the Cartesian product $X \times Y$, of the sets X and Y, equipped with the structure

$$d((x,y),(x',y')) = a(x,x') \wedge b(y,y'),$$

for all $(x,y),(x',y') \in X \times Y$, which is in fact a V-category.
More important to us is, however, the structure $X \otimes Y = (X \times Y, a \otimes b)$, where, for all $(x,y),(x',y') \in X \times Y$,

$$(a \otimes b)((x,y),(x',y')) = a(x,x') \otimes b(y,y').$$

This tensor product $\otimes$ on $\mathsf{V}\text{-}\mathsf{Cat}$ is associative and commutative, and has $1 = (1,k)$ (with a singleton set $1 = \{*\}$ and $k(*,*) = k$) as neutral object. Note that in general $1 = (1,k)$ must be distinguished from the terminal object $(1,\top)$ in $\mathsf{V}\text{-}\mathsf{Cat}$. What makes this structure more interesting is the fact that, unlike $X \times -$, the functor $X \otimes - : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$ has a right adjoint $(\ \ )^X : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$, such that, for any V-category (Y,b),

$$Y^X = (\mathsf{V}\text{-}\mathsf{Cat}(X,Y), d),$$

with $d(f,g) = \bigwedge\limits_{x\in X} b(f(x), g(x))$, for any $f, g \in \mathsf{V}\text{-}\mathsf{Cat}(X,Y)$.
Thus $\mathsf{V}\text{-}\mathsf{Cat}$ is a closed category and there is a natural bijection sending each f in $\mathsf{V}\text{-}\mathsf{Cat}(Z, Y^X)$ to an h in $\mathsf{V}\text{-}\mathsf{Cat}(Z \otimes X, Y)$. Denoting by c the structure of Z, we obtain the following equivalence:

$$c(z,z') \leq b(f(z)(x), f(z')(x))$$

$$\Leftrightarrow c(z,z') \otimes a(x,x') \leq b(h(z,x), h(z',x')),$$

for all $x, x' \in X$ and all $z, z' \in Z$.
The functor $(-)^{\mathrm{op}}$ preserves the tensor product of categories and the neutral category:

$$(X \otimes Y)^{\mathrm{op}} = X^{\mathrm{op}} \otimes Y^{\mathrm{op}}, \qquad\qquad 1^{\mathrm{op}} = 1$$

and $(Y^X)^{\mathrm{op}} \simeq (Y^{\mathrm{op}})^{X^{\mathrm{op}}}$.
In the sequel we will pay particular attention to the V-category $\mathsf{V}^{X^{\mathrm{op}}}$ where $\mathsf{V} = (\mathsf{V}, \mathsf{hom})$.

The following lemma will be useful to work in this category.

Lemma 2.5. *For any u, v in V and any x and y in a V-category (X, a),*

$$\bigwedge_{x' \in X} \mathsf{hom}(u \otimes a(x', x), v \otimes a(x', y)) = \mathsf{hom}(u, v \otimes a(x, y)).$$

Proof. Firstly

$$\begin{aligned} \bigwedge_{x' \in X} \mathsf{hom}(u \otimes a(x', x), v \otimes a(x', y)) &\leq \mathsf{hom}(u \otimes a(x, x), v \otimes a(x, y)) \\ &\leq \mathsf{hom}(u, v \otimes a(x, y)), \end{aligned}$$

for all $u, v \in \mathsf{V}$ and all $x, y \in X$. It remains to prove that, for all $u, v \in \mathsf{V}$ and all $x, y \in X$,

$$\mathsf{hom}(u, v \otimes a(x, y)) \leq \bigwedge_{x' \in X} \mathsf{hom}(u \otimes a(x', x), v \otimes a(x', y))$$

which is equivalent to prove that, for all $x' \in X$,

$$\begin{aligned} &\mathsf{hom}(u, v \otimes a(x, y) \leq \mathsf{hom}(u \otimes a(x', x), v \otimes a(x', y)) \\ \Leftrightarrow &u \otimes a(x', x) \otimes \mathsf{hom}(u, v \otimes a(x, y)) \leq v \otimes a(x', y) \\ \Leftrightarrow &a(x', x) \otimes v \otimes a(x, y) \leq v \otimes a(x', y), \end{aligned}$$

that is true due to the commutativity of $\otimes$ and the transitivity of a in X. □

Examples 2.6. 1. For $\mathsf{V} = 2$, a V-category is just a set equipped with a reflexive and transitive relation, and a V-functor is a monotone map:

$$x \leq x' \Rightarrow f(x) \leq f(x'),$$

for all $x, x' \in X$. Hence, V-Cat is the category Ord of (pre)ordered sets and monotone maps. We do not assume an ordered set to be antisymmetric, and therefore we call the objects of Ord simply ordered sets. Consequently, many notions of order theory such as suprema or infima are only unique up to equivalence $\simeq$, where $x \simeq x'$ if $x \leq x'$ and $x' \leq x$.

2. For $\mathsf{V} = [0, +\infty]$, a V-category structure is a map $a : X \times X \to [0, +\infty]$ which satisfies the conditions

$$0 \geq a(x, x) \qquad \text{and} \qquad a(x, y) + a(y, z) \geq a(x, z),$$

for all $x, y, z \in X$; and a V-functor is a non-expansive map:

$$a(x, x') \geq b(f(x), f(x')),$$

for all $x, x' \in X$. Hence, V-Cat is the category Met of (pre)metric spaces and non-expansive maps. However, in the sequel we follow the nomenclature of [Law73] and call the objects of Met simply metric spaces, then a "classical" metric space becomes a *separated* ($d(x, y) = 0 = d(y, x)$ implies $x = y$), *symmetric* ($d(x, y) = d(y, x)$) and *finitary* ($d(x, y) < +\infty$) metric space.

3. For $\mathsf{V} = \Delta$, a V-category is set equipped with a structure $a : X \times X \to \Delta$ such that

$$1 \leq a(x, x)(t) \qquad \text{and} \qquad \bigvee_{q+r \leq t} a(x, y)(q) \cdot a(y, z)(r) \leq a(x, z)(t)$$

for any $x, y, z \in X$ and $t \geq 0$. A Δ-functor $f : X \to Y$ is a map between the Δ-categories (X, a) and (Y, b) such that

$$a(x, x')(t) \leq b(f(x), f(x'))(t),$$

for all $x, x' \in X$ and $t \geq 0$. This example will be discussed in more depth in Section 2.6.

Every lax morphism of quantales $F : \mathsf{V} \to \mathsf{W}$ induces a functor $F : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{W}\text{-}\mathsf{Cat}$ which sends a V-category $X = (X, a)$ to the W-category $FX = (X, F \cdot a)$ with the same underlying set X and with the W-categorical structure given by the composite

$$X \times X \xrightarrow{a} \mathsf{V} \xrightarrow{F} \mathsf{W};$$

and sends a V-functor $f : (X, a) \to (Y, b)$ to the W-functor $Ff = f : (X, F \cdot a) \to (Y, F \cdot b)$. If the monotone map $F : \mathsf{V} \to \mathsf{W}$ happens to have an adjoint $G : \mathsf{W} \to \mathsf{V}$ which is also a lax morphism of quantales, then the induced functor $G : \mathsf{W}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$ is adjoint to $F : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{W}\text{-}\mathsf{Cat}$. In particular, when $F : \mathsf{V} \to \mathsf{W}$ and $G : \mathsf{W} \to \mathsf{V}$ are inverse to each other, then they induce an isomorphism between $\mathsf{V}\text{-}\mathsf{Cat}$ and $\mathsf{W}\text{-}\mathsf{Cat}$.

Examples 2.7. 1. For the (lax) morphisms of quantales considered in Example 1.15 (1) between the quantales $\mathsf{2}$ and $[0, +\infty]$, the functor $I : \mathsf{Ord} \to \mathsf{Met}$ interprets an order relation $\leq$ on a set X as the metric

$$d(x,y) = \begin{cases} 0 & \text{if } x \leq y, \\ +\infty & \text{else.} \end{cases}$$

The functor $I : \mathsf{Ord} \to \mathsf{Met}$ has a left adjoint $O : \mathsf{Met} \to \mathsf{Ord}$ which takes a metric d on X to the order relation

$$x \leq y \quad \text{if} \quad d(x,y) < +\infty,$$

and a right adjoint $P : \mathsf{Met} \to \mathsf{Ord}$ which sends a metric d on X to the order relation

$$x \leq y \quad \text{if} \quad 0 \geqslant d(x,y).$$

2. Finally, recalling Example 1.15 (2), we find that $\mathsf{Met} \simeq [0,+\infty]\text{-}\mathsf{Cat}$ and $[0,1]\text{-}\mathsf{Cat}$ are isomorphic.

2.3 V-Distributors

A V-relation $\varphi : X {\longrightarrow\!\!\!\!\!\!\mapsto} Y$ between V-categories $X = (X,a)$ and $Y = (Y,b)$ is a V-*distributor* (or V-module or V-profunctor) if $\varphi \cdot a \leq \varphi$ and $b \cdot \varphi \leq \varphi$; that is, if,

$$a(x,x') \otimes \varphi(x',y) \leq \varphi(x,y) \qquad \text{and} \qquad \varphi(x,y') \otimes b(y',y) \leq \varphi(x,y),$$

for all $x, x' \in X$ and $y, y' \in Y$. The usual notation for a V-distributor φ from X to Y is $\varphi : X {\circ\!\!\!\longrightarrow} Y$. The identity distributor on (X,a) is actually a because, by definition, $\varphi \cdot a \leq \varphi$ and, since $1_X \leq a$, we also have $\varphi \leq \varphi \cdot a$. By composing V-distributors as if they were V-relations, one obtains the category $\mathsf{V}\text{-}\mathsf{Dist}$ whose objects are V-categories and morphisms are V-distributors. Since the morphisms are V-relations, the set $\mathsf{V}\text{-}\mathsf{Dist}(X,Y)$, of V-distributors from X to Y, is a complete ordered set where the supremum of a family $\varphi_i : X {\circ\!\!\!\longrightarrow} Y$, $i \in I$, is calculated pointwise.

Examples 2.8. 1. A $\mathsf{2}$-relation $\varphi : X {\circ\!\!\!\longrightarrow} Y$ is a $\mathsf{2}$-distributor if, for all $x, x' \in X$ and all $y, y' \in Y$,

$$(x' \leq x \text{ and } \varphi(x,y) \Rightarrow \varphi(x',y)) \text{ and } (\varphi(x,y) \text{ and } y \leq y' \Rightarrow \varphi(x,y')).$$

In particular, $\varphi : 1 {\circ\!\!\!\longrightarrow} X$ is a $\mathsf{2}$-distributor if, for all $x, x' \in X$, $\varphi(x)$ and $x \leq x'$ implies $\varphi(x')$ and $\psi : X {\circ\!\!\!\longrightarrow} 1$ is a $\mathsf{2}$-distributor if, for all $x, x' \in X$, $x \leq x'$ and $\psi(x')$ implies $\psi(x)$.

2. A $[0, +\infty]$-distributor $\varphi : X \xrightarrow{\circ} Y$ is a $[0, +\infty]$-relation satisfying, for all $x, x' \in X$ and $y, y' \in Y$,

$$a(x, x') + \varphi(x'y) \geq \varphi(x, y) \text{ and } \varphi(x, y') + b(y', y) \geq \varphi(x, y).$$

As a consequence, $\varphi : 1 \xrightarrow{\circ} X$ is a $[0, +\infty]$-distributor if $\varphi(x) + a(x, x') \geq \varphi(x')$, for all $x, x' \in X$, and $\psi : X \xrightarrow{\circ} 1$ is a $[0, +\infty]$-distributor if $a(x, x') + \psi(x') \geq \psi(x)$, for all $x, x' \in X$.

3. A Δ-distributor $\varphi : X \xrightarrow{\circ} Y$ is a Δ-relation satisfying

$$\bigvee_{r+s \leq t} a(x, x')(r) \cdot \varphi(x', y)(s) \leq \varphi(x, y)(t)$$

and

$$\bigvee_{r+s \leq t} \varphi(x, y')(r) \cdot b(y', y)(s) \leq \varphi(x, y)(t),$$

for all $x, x' \in X$, $y, y' \in Y$ and $t \geq 0$. A Δ-relation $\varphi : 1 \relbar\!\!\!+\!\!\!\rightarrow X$ is a Δ-distributor if, for all $x, x' \in X$ and all $t > 0$,

$$\bigvee_{r+s \leq t} \varphi(x)(r) \cdot a(x, x')(s) \leq \varphi(x')(t),$$

and $\psi : X \relbar\!\!\!+\!\!\!\rightarrow 1$ is a Δ-distributor if for all $x, x' \in X$ and all $t > 0$,

$$\bigvee_{r+s \leq t} a(x, x')(r) \cdot \psi(x')(s) \leq \psi(x)(t).$$

Since composition of V-distributors preserves suprema, both monotone maps $- \cdot \varphi$ and $\varphi \cdot -$ (where $\varphi : X \xrightarrow{\circ} Y$) have a right adjoint in Ord. A right adjoint $- \bullet\!\!- \varphi$ of $- \cdot \varphi$ must give, for each $\psi : X \xrightarrow{\circ} Z$, the largest V-distributor of type $Y \xrightarrow{\circ} Z$ whose composite with φ is less or equal then ψ,

$$\begin{tikzcd}
X \arrow[r, "\psi", "\circ"'] \arrow[d, "\varphi"', "\circ"] \arrow[dr, phantom, "\leq"] & Z \\
Y \arrow[ur, dotted, "\circ"'] &
\end{tikzcd}$$

then $\psi \bullet\!\!- \varphi$ is the *extension of ψ along φ*. Explicitly, for all $y \in Y$ and all $z \in Z$,

$$\psi \bullet\!\!- \varphi(y, z) = \bigwedge_{x \in X} \mathsf{hom}(\varphi(x, y), \psi(x, z)).$$

Similarly, a right adjoint $\varphi \multimapdot -$ of $\varphi \cdot -$ must give, for each $\psi : Z \mathrel{-\!\!\circ\!\!\!\to} Y$, the largest V-distributor of type $Z \mathrel{-\!\!\circ\!\!\!\to} X$ whose composite with φ is less or equal then ψ.

$$\begin{tikzcd} Y & Z \arrow[l, "\psi"', "\circ" marking] \arrow[dl, dotted, "\circ" marking] \arrow[dl, phantom, shift right=3, "\leq"] \\ X \arrow[u, "\varphi", "\circ" marking] & \end{tikzcd}$$

The V-distributor $\varphi \multimapdot \psi$ is the *lifting of* ψ *along* φ, and can be calculated as

$$\varphi \multimapdot \psi(z,x) = \bigwedge_{y \in Y} \mathsf{hom}(\varphi(x,y), \psi(z,y)),$$

for all $z \in Z$ and all $x \in X$.
There is an embedding $\mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{V}\text{-}\mathsf{Dist}$ that takes a set X to the V-category $(X, 1_X)$ and the functor $(-)^{\mathrm{op}}$ can be extended to $(-)^{\mathrm{op}} : \mathsf{V}\text{-}\mathsf{Dist} \to \mathsf{V}\text{-}\mathsf{Dist}^{\mathrm{op}}$, that sends $\varphi : X \mathrel{-\!\!\circ\!\!\!\to} Y$ to $\varphi^\circ : Y^{\mathrm{op}} \mathrel{-\!\!\circ\!\!\!\to} X^{\mathrm{op}}$.
The following result is a generalization of Proposition 1.4 for an arbitrary quantale V:

Theorem 2.9. *For* V*-categories* X *and* Y *and for a* V*-relation* $\varphi : X \mathrel{-\!\!\!\mapstochar\!\!\to} Y$ *we have*

$$\varphi : X \mathrel{-\!\!\circ\!\!\!\to} Y \text{ in } \mathsf{V}\text{-}\mathsf{Dist} \Leftrightarrow \varphi : X^{\mathrm{op}} \otimes Y \to \mathsf{V} \text{ in } \mathsf{V}\text{-}\mathsf{Cat}.$$

Proof. Suppose that (X, a) and (Y, b) are V-categories and let $\varphi : X \mathrel{-\!\!\circ\!\!\!\to} Y$ be a V-distributor. For any $x, x' \in X$ and $y, y' \in Y$,

$$\begin{aligned} \varphi(x,y) \otimes a(x',x) \otimes b(y,y') &\leq \varphi(x',y) \otimes b(y,y') \\ &\leq \varphi(x',y'). \end{aligned}$$

Hence

$$a(x',x) \otimes b(y,y') \leq \mathsf{hom}(\varphi(x,y), \varphi(x',y')).$$

If $\varphi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ is a V-functor, then for any $x, x' \in X$ and $y \in Y$,

$$a(x',x) \leq a(x',x) \otimes b(y,y) \leq \mathsf{hom}(\varphi(x,y), \varphi(x',y)),$$

and therefore

$$a(x',x) \otimes \varphi(x,y) \leq \varphi(x',y).$$

The proof of $b \cdot \varphi \leq \varphi$ is similar. $\square$

Thus we can view a V-distributor $\varphi : X \mathrel{-\!\!\circ\!\!\!\to} Y$ as V-functor $\varphi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ or as

$\varphi : Y \to \mathsf{V}^{X^{\mathrm{op}}}$. Due to the importance of the V-category $\mathsf{V}^{X^{\mathrm{op}}}$ we will denote it by $\hat{X}$ and its structure by $\hat{a}$. A consequence of Theorem 2.9 is that the set of V-functors $\hat{X}$ can be identified with $\mathsf{V}\text{-}\mathsf{Dist}(X, 1)$ and under this identification

$$\hat{a}(\psi, \psi') = \bigwedge_{x \in X} \mathsf{hom}(\psi(x), \psi'(x)) = \psi' \multimapdotinv \psi,$$

for all $\psi, \psi' \in \mathsf{V}\text{-}\mathsf{Dist}(X, 1)$.
Finally, we note that every morphism of quantales $F : \mathsf{V} \to \mathsf{W}$ also induces a functor $F : \mathsf{V}\text{-}\mathsf{Dist} \to \mathsf{W}\text{-}\mathsf{Dist}$. Here, for a V-distributor $\varphi : (X, a) \,{\circ\!\!\!\rightarrow}\, (Y, b)$, $F\varphi$ is the W-distributor of type $(X, F \cdot a) \,{\circ\!\!\!\rightarrow}\, (Y, F \cdot b)$ given by the composite

$$X \times Y \xrightarrow{\varphi} \mathsf{V} \xrightarrow{F} \mathsf{W}.$$

Furthermore, F is even locally monotone meaning here that $\varphi \leq \varphi'$ implies $F\varphi \leq F\varphi'$, and, therefore, one has $F\varphi \dashv F\psi$ in $\mathsf{W}\text{-}\mathsf{Dist}$ for every adjunction $\varphi \dashv \psi$ in $\mathsf{V}\text{-}\mathsf{Dist}$.

2.4 Functors between V-Cat and V-Dist

A V-functor gives rise to two V-distributors through the functors:

- $(\ \ )_* : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Dist}$, that has no action on objects and to each V-functor $f : (X, a) \to (Y, b)$ associates the V-distributor $f_* = b \cdot f$ such that

  $$f_*(x, y) = b(f(x), y),$$

  for any $x \in X$ and $y \in Y$.

- $(\ \ )^* : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Dist}^{\mathrm{op}}$, that has no action on objects and to each V-functor $f : (X, a) \to (Y, b)$ associates the V-distributor $f^* = f^\circ \cdot b$ such that

  $$f^*(y, x) = b(y, f(x)),$$

  for any $x \in X$ and $y \in Y$.

Note that, for any map $f : (X, a) \to (Y, b)$, f_* (respectively f^*) is always a V-relation; we obtain a V-distributor if and only if:

$$\begin{array}{rrclcrcl} & f_* \cdot a & \leq & f_* & \text{and} & b \cdot f_* & \leq & f_* \\ \Leftrightarrow & b \cdot f \cdot a & \leq & b \cdot f & \text{and} & b \cdot b \cdot f & \leq & b \cdot f \\ \Leftrightarrow & \multicolumn{7}{c}{f \cdot a \leq b \cdot f \cdot a \leq b \cdot f} \end{array}$$

or, equivalently, if and only if f is a V-functor.
Any element of X can be seen as a V-functor of the type $x : 1 \to X$. For this particular case, $x_* = a(x, -)$ and $x^* = a(-, x)$.
For later usage we record the calculation rules

$$f^* \cdot \varphi(z, x) = \varphi(z, f(x)), \qquad\qquad \psi \cdot f_*(x, z) = \psi(f(x), z);$$

for V-distributors $\varphi : Z \longrightarrow\!\!\!\!\!\!\circ\;\; Y$ and $\psi : Y \longrightarrow\!\!\!\!\!\!\circ\;\; Z$.

Definition 2.10. A V-functor $f : (X, a) \to (Y, b)$ is *fully faithful* if $a = f^* \cdot f_*$ and f is *fully dense* if $f_* \cdot f^* = b$.

In pointwise notation, a V-functor $f : (X, a) \to (Y, b)$ is fully faithful if

$$a(x, x') = b(f(x), f(x')),$$

for all $x, x' \in X$. Hence a fully faithful V-functor preserves and reflects "distances". A V-functor $f : (X, a) \to (Y, b)$ is fully dense if

$$\bigvee_{x \in X} b(y, f(x)) \otimes b(f(x), y') = b(y, y'),$$

for all $y, y' \in Y$. If $f : X \to Y$ is a V-functor satisfying $b(y, y) \leq \bigvee_{x \in X} b(y, f(x)) \otimes b(f(x), y)$, for all $y \in Y$, then,

$$\begin{aligned} b(y, y') &\leq b(y, y) \otimes b(y, y') \\ &\leq \bigvee_{x \in X} b(y, f(x)) \otimes b(f(x), y) \otimes b(y, y') \\ &\leq \bigvee_{x \in X} b(y, f(x)) \otimes b(f(x), y'). \end{aligned}$$

Thus,

Lemma 2.11. *A V-functor $f : X \to Y$ is fully dense if, for all $y \in Y$,*

$$b(y, y) \leq \bigvee_{x \in X} b(y, f(x)) \otimes b(f(x), y).$$

Proposition 2.12. *Given a V-functor $f : (X, a) \to (Y, b)$ the following statements are equivalent:*

1. *f is fully faithful;*
2. *$f^* \cdot f_* \leq a$;*

3. $f^\circ \cdot b \cdot f \leq a$.

Proof. If $f : (X, a) \to (Y, b)$ is fully faithful it is obvious that $f^* \cdot f_* \leq a$ which is equivalent to $f^\circ \cdot b \cdot f \leq a$, since $b \cdot b = b$. This last inequality and the fact that $f \dashv f^\circ$ implies that f is fully faithful. □

Lemma 2.13. *The* V*-functor* $1_X \otimes i : X \otimes Y \to X \otimes Z$ *is fully faithful whenever* $i : Y \to Z$ *is fully faithful and it is fully dense whenever* $i : Y \to Z$ *is fully dense.*

Proof. Denote the structure on the V-categories X, Y and Z by, respectively, a, b and c. If i is fully faithful then, for all $y, y' \in Y$,

$$b(y, y') = c(i(y), i(y')).$$

Thus, for all $x, x' \in X$ and $y, y' \in Y$,

$$\begin{aligned} a \otimes b((x, y), (x', y')) &= a(x, x') \otimes b(y, y') \\ &= a(x, x') \otimes c(i(y), i(y')). \end{aligned}$$

Hence,

$$a \otimes b((x, y), (x', y')) = a \otimes c(1_X \otimes i(x, y), 1_X \otimes i(x', y')),$$

for all $x, x' \in X$ and $y, y' \in Y$, which means that $1_X \otimes i$ is also fully faithful.
Suppose that i is fully dense, that is, for all $z \in Z$,

$$\bigvee_{y \in Y} c(z, i(y)) \otimes c(i(y), z) = c(z, z).$$

Then, for all $(x, z)) \in X \times Z$,

$$\begin{aligned} &\bigvee_{(x', y') \in X \times Y} (a \otimes c)((x, z), 1_X \otimes i(x', y')) \otimes (a \otimes c)(1_X \otimes i(x', y'), (x, z)) \\ =&\bigvee_{(x', y') \in X \times Y} (a \otimes c)((x, z), (x', i(y'))) \otimes (a \otimes c)((x', i(y')), (x, z)) \\ =&\bigvee_{(x', y') \in X \times Y} a(x, x') \otimes c(z, i(y')) \otimes a(x', x) \otimes c(i(y'), z) \\ =&a(x, x) \otimes c(z, z). \end{aligned}$$

Thus $1_X \otimes i$ is also fully dense. □

Lemma 2.14. *The* V*-functor* $1_X \times i : X \times Y \to X \times Z$ *is fully faithful whenever* $i : Y \to Z$ *is fully faithful.*

Proof. Suppose that $i : (Y, b) \to (Z, c)$ is a fully faithful V-functor. Thus, for all $y, y' \in Y$, $b(y, y') = c(i(y), i(y'))$. Let $(x, y), (x', y') \in X \times Y$,

$$\begin{aligned} a \wedge b((x,y),(x',y')) &= a(x,x') \wedge b(y,y') \\ &= a(1_X(x), 1_X(x')) \wedge c(i(y), i(y')) \\ &= a \wedge c((1_X(x), i(y)), (1_X(x'), i(y'))) \\ &= a \wedge c(1_X \times i(x,y), 1_X \times i(x',y')). \end{aligned}$$

Then $1_X \times i$ is fully faithful. $\square$

Obviously, Lemmas 2.13 and 2.14 are also valid changing the order of the factors in each of the products.

Examples 2.15. 1. For $\mathsf{V} = 2$, a monotone map $f : X \to Y$ is fully faithful if, for all $x, x' \in X$, $x \leq x' \iff f(x) \leq f(x')$, and it is fully dense if, for all $y \in Y$ there is some $x \in X$ such that $y \leq f(x) \leq y$. If the order $\leq$ in X is antisymmetric then a fully dense monotone map is a surjective map.

2. For the quantale $[0, +\infty]$, a fully faithful non-expansive map is a map that preserves and reflects distances and a non-expansive map $f : (X, a) \to (Y, b)$ is fully dense if, for any $y \in Y$,

$$\inf_{x \in X} b(y, f(x)) + b(f(x), y) = 0.$$

3. A Δ-functor $f : (X, a) \to (Y, b)$ is fully faithful whenever, for any $x, x' \in X$, $a(x, x')$ and $b(f(x), f(x'))$ represent the same map $[0, +\infty] \to [0, 1]$ in Δ and it is fully dense if, for any $y \in Y$ and any $t \in [0, +\infty]$,

$$\bigvee_{x \in X} \bigvee_{q+r \leq t} b(y, f(x))(q) \cdot b(f(x), y)(r) = 1.$$

Finally note that a functor $F : \mathsf{V}\text{-}\mathsf{Dist} \to \mathsf{W}\text{-}\mathsf{Dist}$, induced by a morphism of quantales, extends $F : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{W}\text{-}\mathsf{Cat}$ in the sense that both diagrams

(2.1)
$$\begin{tikzcd} \mathsf{V}\text{-}\mathsf{Dist} \arrow[r, "F"] & \mathsf{W}\text{-}\mathsf{Dist} \\ \mathsf{V}\text{-}\mathsf{Cat} \arrow[u, "(-)_*"] \arrow[r, "F"'] & \mathsf{W}\text{-}\mathsf{Cat} \arrow[u, "(-)_*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{V}\text{-}\mathsf{Dist}^{\mathrm{op}} \arrow[r, "F^{\mathrm{op}}"] & \mathsf{W}\text{-}\mathsf{Dist}^{\mathrm{op}} \\ \mathsf{V}\text{-}\mathsf{Cat} \arrow[u, "(-)^*"] \arrow[r, "F"'] & \mathsf{W}\text{-}\mathsf{Cat} \arrow[u, "(-)^*"'] \end{tikzcd}$$

commute.

2.5 Adjunctions in V-Dist

Since V-Dist has ordered hom-sets, it is possible to talk about adjunctions between of V-distributors. Given $\varphi : (X, a) \multimap (Y, b)$ and $\psi : (Y, b) \multimap (X, a)$:

$$\varphi \dashv \psi \Leftrightarrow a \leq \psi \cdot \varphi \text{ and } \varphi \cdot \psi \leq b$$

In pointwise notation, $\varphi \dashv \psi$ if, for all $x, x' \in X$ and all $y, y' \in Y$,

$$a(x, x') \leq \bigvee_{y \in Y} \varphi(x, y) \otimes \psi(y, x')$$

and

$$\psi(y, x) \otimes \varphi(x, y') \leq b(y, y').$$

When the V-distributors are of the form $\varphi : 1 \multimap X$ and $\psi : X \multimap 1$, then $\varphi \dashv \psi$ if, for all $x, x' \in X$,

$$k \leq \bigvee_{x \in X} \varphi(x) \otimes \psi(x) \qquad \text{and} \qquad \psi(x) \otimes \varphi(x') \leq a(x, x').$$

As with V-functors in Section 2.2, adjoint V-distributors determine each other meaning that $\varphi \dashv \psi$ and $\varphi \dashv \psi'$ imply $\psi = \psi'$, as well as $\varphi \dashv \psi$ and $\varphi' \dashv \psi$ imply $\varphi = \varphi'$. Therefore, one says that φ is left adjoint whenever $\varphi \dashv \psi$ for some (unique) ψ, and that ψ is right adjoint if $\varphi \dashv \psi$ for some (unique) φ.

Examples 2.16. 1. Let $\mathsf{V} = 2$. The 2-distributor $\varphi : X \multimap Y$ is left adjoint to $\psi : Y \multimap X$ if and only if

$$x \leq x' \Rightarrow \exists y \in Y : \varphi(x, y) \text{ and } \psi(y, x')$$

and

$$\psi(y, x) \text{ and } \varphi(x, y') \Rightarrow y \leq y',$$

for all $x, x' \in X$ and all $y, y' \in Y$. The special case of an adjunction $\varphi \dashv \psi$ where $\varphi : 1 \multimap X$ and $\psi : X \multimap 1$ has the following characterization:

$$\exists x \in X : \varphi(x) \text{ and } \psi(x) \quad \text{and} \quad \forall x, x' \in X, \psi(x) \text{ and } \varphi(x') \Rightarrow x \leq x'.$$

The first condition fleshes out the existence of an element of X that is simultaneously in φ and in ψ, while the second states that if x is in ψ and x' is in φ then $x \leq x'$. Therefore, one concludes that there is $x \in X$ such that $\varphi = x_*$ and

$\psi = x^*$.

2. An adjunction $\varphi \dashv \psi : Y \to\!\!\!\!\!\!\circ\ X$ in $[0,+\infty]$-Dist is characterised by

$$a(x,x') \geq \inf_{y\in Y} \varphi(x,y) + \psi(y,x') \quad \text{and} \quad \psi(y,x) + \varphi(x,y') \geq b(y,y'),$$

for all $x, x' \in X$ and all $y, y' \in Y$. If $\varphi : 1 \to\!\!\!\!\!\!\circ\ X$ and $\psi : X \to\!\!\!\!\!\!\circ\ 1$ then $\varphi \dashv \psi$ if and only if

$$\inf_{x\in X} \varphi(x) + \psi(x) = 0 \quad \text{and} \quad \forall x, x' \in X, a(x,x') \leq \psi(x) + \varphi(x').$$

3. In $\mathsf{V} = \Delta$, given Δ-distributors $\varphi : X \to\!\!\!\!\!\!\circ\ Y$ and $\psi : Y \to\!\!\!\!\!\!\circ\ X$, we have $\varphi \dashv \psi$ if and only if

$$a(x,x')(t) \leq \bigvee_{y\in Y} \bigvee_{r+s\leq t} \varphi(x,y)(r) \cdot \psi(y,x')(s)$$

and

$$\bigvee_{r+s\leq t} \psi(y,x)(r) \cdot \varphi(x,y')(s) \leq b(y,y')(t),$$

for all $x, x' \in X$, all $y, y' \in Y$ and all $t \geq 0$. In particular, if $\varphi : 1 \to\!\!\!\!\!\!\circ\ X$ and $\psi : X \to\!\!\!\!\!\!\circ\ 1$ then $\varphi \dashv \psi$ if and only if, for all $x, x' \in X$ and for all $t \geq 0$,

$$\bigvee_{x\in X} \varphi(x) \otimes \psi(x)(t) = 1 \quad \text{and} \quad \psi(x) \otimes \varphi(x')(t) \leq a(x,x')(t),$$

or, equivalently,

$$\bigvee_{x\in X} \bigvee_{r+s\leq t} \varphi(x)(r) \cdot \psi(x)(s) = 1$$

and

$$\bigvee_{r+s\leq t} \psi(x)(r) \cdot \varphi(x')(s) \leq a(x,x')(t).$$

The following result is analogous to Lemma 2.1 in V-Dist and will be extremely useful for calculating with adjoints.

Lemma 2.17. *Let $\varphi, \varphi' : X \to\!\!\!\!\!\!\circ\ Y$ and $\psi, \psi' : Y \to\!\!\!\!\!\!\circ\ X$ be V-distributors with $\varphi \dashv \psi$, $\varphi' \dashv \psi'$, $\varphi \leq \varphi'$ and $\psi \leq \psi'$. Then $\varphi = \varphi'$ and $\psi = \psi'$.*

Proof. By unicity of adjoints, it is enough to show that $\varphi = \varphi'$, that is, $\varphi \geq \varphi'$. Hence, we calculate $\varphi \geq \varphi' \cdot \psi' \cdot \varphi \geq \varphi' \cdot \psi \cdot \varphi \geq \varphi'$. □

Given a V-functor $f : (X,a) \to (Y,b)$, the V-distributors f_* and f^* are adjoints: $f_* \dashv f^*$.

In fact, since f is a V-functor,

$$f^* \cdot f_* = f^\circ \cdot b \cdot b \cdot f = f^\circ \cdot b \cdot f \geq f^\circ \cdot f \cdot a \geq a$$

and

$$f_* \cdot f^* = b \cdot f \cdot f^\circ \cdot b \leq b \cdot b = b.$$

The V-distributor identity $(1_X)^* = a : X {\longrightarrow\!\!\!\!\!\circ\,} X$ can be seen as the V-functor $X^{\mathrm{op}} \otimes X \to \mathsf{V}$ or $X \to \hat{X}$. Then:

$$\begin{aligned} a : X &\to \hat{X} \\ x &\mapsto a(-,x) \end{aligned}$$

is the *Yoneda* V*-functor* $\mathcal{Y}_X$. Note that $\mathcal{Y}_X(x) = a(-,x) = x^*$, thinking of x as the V-functor $x : 1 \to X$.

Lemma 2.18 (Yoneda Lemma)**.** *Given $x \in X$ and $f \in \hat{X}$, $\hat{a}(x^*, f) = f(x)$.*

Proof. Let $x \in X$ and $f \in \hat{X}$; since f can be viewed as a V-distributor $f : X {\longrightarrow\!\!\!\!\!\circ\,} 1$ we know that

$$f \cdot a \leq f \Leftrightarrow \bigvee_{x \in X} a(x', x) \otimes f(x) \leq f(x'),$$

for all $x \in X$. Thus, for all $x, x' \in X$,

$$\begin{aligned} & a(x', x) \otimes f(x) \leq f(x') \\ \Leftrightarrow & f(x) \leq \mathsf{hom}(a(x', x), f(x')). \end{aligned}$$

Therefore

$$\begin{aligned} & f(x) \leq \bigwedge_{x' \in X} \mathsf{hom}(a(x', x), f(x')) \\ \Leftrightarrow & f(x) \leq \hat{a}(a(-,x), f), \end{aligned}$$

for all $x \in X$. Furthermore, since, for all $x \in X$,

$$k \otimes \mathsf{hom}(a(x,x), f(x)) \leq a(x,x) \otimes \mathsf{hom}(a(x,x), f(x)) \leq f(x),$$

then,

$$\hat{a}(x^*, f) \leq \mathsf{hom}(a(x,x), f(x)) \leq f(x).$$ □

Lemma 2.19. *The Yoneda* V*-functor is fully faithful.*

Proof. By Yoneda Lemma, for all $x, x' \in X$,

$$\hat{a}(a(-,x), a(-,x')) = a(x,x'). \qquad \square$$

Lemma 2.20. *If $\varphi \dashv \psi : X \to\!\!\!\!\!\circ\!\!\to 1$ then $\varphi = a \bullet\!\!- \psi$ and $\psi = \varphi -\!\!\bullet a$.*

Proof. Suppose that $\varphi \dashv \psi$ and that $\gamma : 1 \to\!\!\!\!\!\circ\!\!\to X$ verifies $\gamma \cdot \psi \leq a$. Thus

$$\gamma \leq \gamma \cdot \psi \cdot \varphi \leq a \cdot \varphi \leq \varphi,$$

which means that φ is the largest V-distributor whose composite with ψ is less or equal a, ie, φ is the extension of a along ψ. The second statement is proven similarly. $\square$

If $\psi : X \to\!\!\!\!\!\circ\!\!\to 1$ has a left adjoint φ then $- \cdot \psi \dashv - \cdot \varphi$. This means that, under these circumstances, $- \cdot \varphi = - \bullet\!\!- \psi$ and, furthermore, $\hat{a}(\psi, \psi') = \psi' \bullet\!\!- \psi = \psi' \cdot \varphi$, for any $\psi' : X \to\!\!\!\!\!\circ\!\!\to 1$. In particular, if $\psi' = x^*$ then, for any $x \in X$,

$$\hat{a}(\psi, x^*) = x^* \cdot \varphi = \varphi(x).$$

For any V-distributor $\psi : X \to\!\!\!\!\!\circ\!\!\to 1$, seen also as an element of $\hat{X}$, the Yoneda Lemma shows that $\psi^* \cdot (\mathcal{Y}_X)_* = \psi$. If, moreover, ψ has a left adjoint φ, then also $\mathcal{Y}_X^* \cdot \psi_* = \hat{a}(\psi, (-)^*) = \varphi$, and therefore $\psi^* \geq \psi \cdot \mathcal{Y}_X^*$ and $\psi_* \geq (\mathcal{Y}_X)_* \cdot \varphi$. Hence, Lemma 2.17 implies

Lemma 2.21. *For every adjunction $(\varphi : 1 \to\!\!\!\!\!\circ\!\!\to X) \dashv (\psi : X \to\!\!\!\!\!\circ\!\!\to 1)$, $\psi^* = \psi \cdot \mathcal{Y}_X^*$ and $\psi_* = (\mathcal{Y}_X)_* \cdot \varphi$.*

2.6 Probabilistic metric spaces

The classical definition of probabilistic metric space [Men42, SS83] considers a set X, a binary operation $* : [0,1] \times [0,1] \to [0,1]$ with neutral element 1, usually called *t-norm*, which makes the interval $[0,1]$ a quantale, and a distance $d : X \times X \times [0,+\infty] \to [0,1]$ subject to

1. $d(x,y,-) : [0,+\infty] \to [0,1]$ is left continuous (that is, $d(x,y,t) = \bigvee_{s<t} d(x,y,s)$, for all t),

2. $d(x,x,t) = 1$ for $t > 0$,

3. $d(x,y,r) * d(y,z,s) \leq d(x,z,r+s)$,

4. $d(x,y,t) = 1 = d(y,x,t)$ for all $t > 0$ implies $x = y$,

5. $d(x, y, t) = d(y, x, t)$ for all t,

6. $d(x, y, +\infty) = 1$,

for all $x, y \in X$ and $r, s \in [0, +\infty]$. One can interpret $d(x, y, t) = u$ as u being the "probability that the distance from x to y is less then t". In the sequel it will be considered that the t-norm defined in $[0, 1]$ is just the usual multiplication. The first condition of this definition, (1), just states that the exponential mate $d : X \times X \to [0, 1]^{[0,+\infty]}$ of d takes values on Δ, and (2) and (3) guarantee that d is reflexive and transitive. Thus a structure satisfying these three conditions is a Δ-category and this is exactly what we will consider to be a probabilistic metric space. Therefore, a probabilistic metric space $X = (X, a)$ satisfies (4) if and only if X is separated, and X satisfies (5) if and only if X is symmetric. Similarly to the nomenclature for metric spaces, we call a Δ-category $X = (X, a)$ *finitary* if X satisfies (6), i.e. if $a(x, y) \in \Delta$ is finite for all $x, y \in X$. Intuitively, (6) states that the probability of the distance from x to y to be finite is equal to 1.
Finally recall that a Δ-functor $f : (X, a) \to (Y, b)$ is a map satisfying

$$a(x, y)(t) \leq b(f(x), f(y))(t) \tag{2.2}$$

for all $x, y \in X$ and $t \in [0, +\infty]$. In other words, the "probability of the distance from x to y is less than t" is less or equal than the "probability of the distance from $f(x)$ to $f(y)$ is less than t". We write $\mathsf{ProbMet}$ for the category of probabilistic metric spaces and maps satisfying (2.2), that is, $\mathsf{ProbMet} \simeq \Delta\text{-}\mathsf{Cat}$.

Remark 2.22. The notion of fuzzy metric space as defined in [GV94] is closely related to the one of probabilistic metric spaces, but in [GV94] it is required that the map $d(x, y, -)$ is continuous. With such a characterisation, the set of distribution functions ceases to be complete for the pointwise order. Hence many nice properties of probabilistic metric spaces are not shared by fuzzy metric spaces, for instance, there exist fuzzy metric spaces which do not admit a L-completion (see [GR02]). We also point out that the notion of a fuzzy metric space of [KM75] is equivalent to the notion of a probabilistic metric spaces (in the classical sense), for any continuous t-norm, as shown in [KM75].

2.7 Comparison with metric spaces

In Examples 1.16 we saw that the quantale $[0, +\infty]$ embeds canonically into Δ via the morphism of quantales

$$I_\infty : [0, +\infty] \to \Delta,\ n \mapsto f_{n,1},$$

and that I_∞ is a morphism of quantales with right adjoint

$$P_\infty : \Delta \to [0, +\infty],\ f \mapsto \inf\{n \in [0, +\infty] : f(n) = 1\}.$$

Since P_∞ does not preserve suprema it is only a lax morphism of quantales. Furthermore, I_∞ has also a left adjoint

$$O_\infty : \Delta \to [0, +\infty],\ f \mapsto \sup\{n \in [0, +\infty] : f(n) = 0\},$$

that is also a morphism of quantales. Then one obtains the chain of adjoint functors

$$\begin{tikzcd} \mathsf{Met} \arrow[rr, "I_\infty" description] \arrow[rr, phantom, bend left=25, "\perp"] \arrow[rr, phantom, bend right=25, "\perp"] & & \mathsf{ProbMet}. \arrow[ll, bend right=40, "O_\infty"'] \arrow[ll, bend left=40, "P_\infty"] \end{tikzcd}$$

between the categories Met and $\mathsf{ProbMet}$.

Chapter 3

L-complete V-categories

In Section 2.5 we have already mentioned that each V-functor $f : (X, a) \to (Y, b)$ induces an adjoint pair $f_* \dashv f^*$ of V-distributors. One of the amazing discoveries of [Law73] is that, in the context of metric spaces, the reverse affirmation (every adjoint pair of V-distributors is induced by a V-functor) is ultimately linked to Cauchy completeness. In fact, in [Law73] it is shown that

Theorem. *A metric space X (viewed as a $[0, +\infty]$-category) is Cauchy complete if and only if every adjunction $\varphi \dashv \psi$, with $\varphi : X \,{-\!\!\circ\!\!\rightarrow}\, Y$ and $\psi : Y \,{-\!\!\circ\!\!\rightarrow}\, X$ in $[0, +\infty]$-Dist, is of the form $f_* \dashv f^*$, for a non-expansive map $f : X \to Y$.*

The aim of this chapter is to establish the conditions that allow the generalisation of this statement to the realm of V-categories. Taking as a starting point the work already done by Lawvere [Law73] in metric spaces as categories enriched in the quantale $[0, +\infty]$ and subsequent developments in [HT10, HR13, Wag94, Tho08, CH08], and others, we will establish conditions that allow the generalisation of results relating Cauchy sequences, convergence of sequences, adjunctions of V-distributors and its representability. At the end of this chapter, notions and results achieved will be compared with the ones in [Fla97], the latter one mainly inspired by the usual analytic description, and applied to the quantale Δ. In this particular context, the work in [GR02] will also be considered.

3.1 Topology in a V-category

In a metric space X, the closure $\overline{A}$ of a subset A of X is the set of those elements of X that are limits of sequences in A. As shown in [Law73], since there is a bijection between equivalence classes of Cauchy sequences and adjunctions of $[0, +\infty]$-distributors, and

convergence is equivalent to representability, a point x is in the closure of A if there is an adjunction $1 \xrightarrow{\circ} A \dashv A \xrightarrow{\circ} 1$ represented by x.
Generalizing for any V-category (X,a) one can define:

Definition 3.1. Let (X,a) be a V-category and $A \subseteq X$. A point $x \in X$ is called a *closure point* of A, written as $x \in \overline{A}$, whenever there is an adjunction $1 \xrightarrow{\circ} A \dashv A \xrightarrow{\circ} 1$ represented by x, that is, $x \in \overline{A}$ if $m^* \cdot x_* \dashv x^* \cdot m_*$, where $m : A \hookrightarrow X$ denotes the full embedding of A in X.

This closure operator was largely studied in [HT10]. Below we recall some important properties whose proofs can be found at [HT10].

Proposition 3.2. *Let A be a V-subcategory of (X,a), $m : A \hookrightarrow X$ the embedding of A in X and $x \in X$. The following assertions are equivalent:*

i) $x \in \overline{A}$;

ii) $a(x,x) \leq \bigvee_{y\in A} a(x,y) \otimes a(y,x)$;

iii) $k \leq \bigvee_{y\in A} a(x,y) \otimes a(y,x)$;

iv) $1_1^* \leq x^* \cdot m_* \cdot m^* \cdot x_*$;

v) $m^* \cdot x_* \dashv x^* \cdot m_*$;

vi) $x_* : 1 \xrightarrow{\circ} X$ *factors trough* $m_* : A \xrightarrow{\circ} X$ *by a map* $\varphi : 1 \xrightarrow{\circ} A$ *in* V-Dist.

By the proposition above, for $x, x' \in \overline{A}$ one has

$$a(x,x') \leq \bigvee_{y\in A} a(x,y) \otimes a(y,x) \otimes a(x,x') \leq \bigvee_{y\in A} a(x,y) \otimes a(y,x') \leq a(x,x'),$$

hence $a(x,x') = \bigvee_{y\in A} a(x,y) \otimes a(y,x')$.

Examples 3.3.
1. For $\mathsf{V} = 2$, a point x of an ordered set X is in $\overline{A}$, with $A \subseteq X$, if there is $y \in A$ such that $x \leq y \leq x$. If the order in X is antisymmetric then every $x \in \overline{A}$ is in A which means that every ordered set is closed.

2. For $\mathsf{V} = [0,+\infty]$, a point x of a metric space X is in $\overline{A}$, with $A \subseteq X$, if

$$\inf_{y\in A} a(x,y) + a(y,x) = 0.$$

3. For $\mathsf{V} = \Delta$, let (X, a) be a Δ-category, $A \subseteq X$ and $x \in X$.

$$\begin{aligned} x \in \overline{A} &\Leftrightarrow k = \bigvee_{y \in A} a(x,y) \otimes a(y,x) \\ &\Leftrightarrow \forall t > 0, 1 = \bigvee_{y \in A} a(x,y) \otimes a(y,x)(t) \\ &\Leftrightarrow \forall t > 0, \forall \epsilon < 1, \exists y \in A : \epsilon \leq a(x,y) \otimes a(y,x)(t). \end{aligned}$$

Proposition 3.4. *Let $f : X \to Y$ be a V-functor, $A, A' \subseteq X$ and $B \subseteq Y$. Then*

i) $A \subseteq \overline{A}$,

ii) $A \subseteq A'$ *implies* $\overline{A} \subseteq \overline{A'}$,

iii) $\overline{\varnothing} = \varnothing$ *and* $\overline{\overline{A}} = \overline{A}$,

iv) $f(\overline{A}) \subseteq \overline{f(A)}$ *and* $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$,

v) $\overline{A \cup A'} = \overline{A} \cup \overline{A'}$ *provided that* $k \leq u \vee v$ *implies* $k \leq u$ *or* $k \leq v$ *for all* $u, v \in \mathsf{V}$.

Furthermore, $\overline{(-)}$ is hereditary, that is, for $A \subseteq Z \subseteq X$ where we consider Z as a V-subcategory of X, $\overline{A}$ calculated in the V-category Z is equal to $\overline{A} \cap Z$ with $\overline{A}$ calculated in the V-category X.

A *closure space* is a set X with a closure operator satisfying (i), (ii) and (iii) of Proposition 3.4. A map between closure spaces is *continuous* if $f(\overline{A}) \subseteq \overline{f(A)}$. We denote the category of closure spaces and continuous maps by Cls.

Corollary 3.5. *The closure operator $\overline{(-)}$ induces a functor $G_{\mathsf{V}} : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{Cls}$. Furthermore, if $k \leq u \vee v$ implies $k \leq u$ or $k \leq v$ for all $u, v \in \mathsf{V}$, then the closure operator $\overline{(-)}$ defines a topology in X and induces a functor $\mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{Top}$.*

Remark 3.6. Note that if $A = \{u \in \mathsf{V} : u \ll k\}$ is directed and $k = \bigvee A$ then $k \leq u \vee v$ implies $k \leq u$ or $k \leq v$ for all $u, v \in \mathsf{V}$, and also $k = \bigvee_{u \ll k} u \otimes u$ ([HR13, Fla97]). Hence, in these conditions, the closure operator is topological.

An important example of a closed V-category is the V-subcategory $\widetilde{X}$ of $\hat{X}$, whose elements are the right adjoint V-distributors of the type $X \to\!\!\!\!\circ\, 1$:

$$\widetilde{X} = \{\psi : X \to\!\!\!\!\circ\, 1 : \psi \text{ is right adjoint }\}.$$

Lemma 3.7. *For any V-category X, $\overline{\mathcal{Y}_X(X)} = \widetilde{X}$.*

Proof. By Lemma 2.20 $\psi : X \mathrel{-\!\!\circ\!\!\rightarrow} 1$ is right adjoint if and only if $a \bullet\!\!- \psi \dashv \psi$,

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "\psi"'] & X \\ 1 \arrow[ur, dotted, "a \bullet\!- \psi"'] & \end{tikzcd}$$

which is equivalent to

$$\begin{aligned} &(a \bullet\!\!- \psi) \cdot \psi \leq a \text{ and } k \leq \psi \cdot (a \bullet\!\!- \psi) \\ \Leftrightarrow &k \leq \bigvee_{x\in X} \bigwedge_{y \in X} \mathsf{hom}(\psi(y), a(y,x)) \otimes \psi(x). \end{aligned}$$

Applying the Yoneda Lemma [2.18],

$$\begin{aligned} &k \leq \bigvee_{x\in X} \hat{a}(\psi, x^*) \otimes \hat{a}(x^*, \psi) \\ \Leftrightarrow &k \leq \bigvee_{x\in X} \hat{a}(\psi, \mathcal{Y}_X(x)) \otimes \hat{a}(\mathcal{Y}_X(x), \psi) \\ \Leftrightarrow &k \leq \bigvee_{y\in \mathcal{Y}_X(X)} \hat{a}(\psi, y) \otimes \hat{a}(y, \psi). \end{aligned}$$

Thus we conclude that $\psi \in \overline{\mathcal{Y}_X(X)}$. □

Finally, let us see which is the action of a functor induced by a morphism of quantales over the closure operator.

Proposition 3.8. *Let $F : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{W}\text{-}\mathsf{Cat}$ be a functor induced by a morphism of quantales. For any V-category (X, a), the map $\lambda_X : G_{\mathsf{V}}(X) \to G_{\mathsf{W}} \cdot F(X)$, such that $\lambda_X(x) = x$, is continuous.*

Proof. If $\overline{A}$ is the closure of a subset A of the V-category (X, a) and $x \in \overline{A}$ then

$$k \leq \bigvee_{y\in A} a(x,y) \otimes a(y,x).$$

Hence

$$k = F(k) \leq F(\bigvee_{y\in A} a(x,y) \otimes a(y,x)) = \bigvee_{y\in A} F \cdot a(x,y) \otimes F \cdot a(y,x),$$

that is, any x in $\overline{A}$ is also in $\overline{F(A)}$, where $\overline{A}$ is the closure of A in the V-category (X, a) and $\overline{F(A)}$ is the closure of A in the W-category $(X, F \cdot a)$. □

Corollary 3.9. *If $F : \mathsf{V} \to \mathsf{W}$ is a morphism of quantales then the diagram*

$$\begin{tikzcd} \mathsf{V}\text{-}\mathsf{Cat} \arrow[rr, "F"] \arrow[dr, "G_{\mathsf{V}}"'] & & \mathsf{W}\text{-}\mathsf{Cat} \arrow[dl, "G_{\mathsf{W}}"] \\ & \mathsf{Cls} & \end{tikzcd}$$

commutes whenever, for any $u, v \in \mathsf{V}$, $F(u) \leq F(v)$ implies $u \leq v$, that is, whenever F is fully faithful.

3.2 Cauchy sequences in a V-category

Let V be a quantale and (X, a) a V-category. For a sequence $s = (x_n)_{n\in\mathbb{N}}$ in (X, a) and $x \in X$, one defines (see [Wag94])

$$\mathsf{Cauchy}_X(s) = \bigvee_{N\in\mathbb{N}} \bigwedge_{m,n\geq N} a(x_m, x_n).$$

Definition 3.10. A sequence $s = (x_n)_{n\in\mathbb{N}}$ in a V-category (X, a) is a Cauchy sequence if

$$k \leq \mathsf{Cauchy}_X(s).$$

In the sequel we will simply write $\mathsf{Cauchy}(s)$ if it is understood from the context which V-category we consider. Note that $\mathsf{Cauchy}_X(s) = \mathsf{Cauchy}_{X^{\mathrm{op}}}(s)$, and $\mathsf{Cauchy}_X(s) = \mathsf{Cauchy}_Y(s)$ for every V-category Y having X as a V-subcategory.

Examples 3.11.

1. In $\mathsf{V} = 2$, a sequence $s = (x_n)_{n\in\mathbb{N}}$ in an ordered set X is Cauchy if

$$\exists N \in \mathbb{N} : \forall n, m \geq N, x_n \leq x_m.$$

   Since this is valid for any $n, m \geq N$, also $x_m \leq x_n$; therefore, if the order relation is antisymmetric, a sequence $s = (x_n)_{n\in\mathbb{N}}$ is Cauchy if it is constant after a certain order.

2. In a metric space a sequence $s = (x_n)_{n\in\mathbb{N}}$ is Cauchy if

$$0 = \inf_{N\in\mathbb{N}} \sup_{n,m\geq N} a(x_n, x_m),$$

   or, equivalently,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, a(x_n, x_m) \leq \epsilon.$$

3. In Δ, $s = (x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, a) precisely if, for all $t > 0$,

$$1 = \bigvee_{N\in\mathbb{N}} \bigwedge_{n,m\geq N} a(x_n, x_m)(t)$$

$$\Leftrightarrow \forall\epsilon < 1, \exists N \in \mathbb{N} : \forall n, m \geq N, \epsilon \leq a(x_n, x_m)(t).$$

Lemma 3.12. *For any sequence $s = (x_n)_{n\in\mathbb{N}}$ in (X, a) and $x \in X$:*

$$\bigwedge_{N\in\mathbb{N}} \bigvee_{n\geq N} a(x_n, x) \otimes \mathsf{Cauchy}(s) \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} a(x_n, x).$$

Proof. Let $s = (x_n)_{n\in\mathbb{N}}$ be a sequence in (X, a) and $x \in X$. Thus,

$$\begin{aligned}
\bigwedge_{N\in\mathbb{N}} \bigvee_{n\geq N} a(x_n, x) \otimes \mathsf{Cauchy}(s) &= \bigwedge_{N\in\mathbb{N}} \bigvee_{n\geq N} a(x_n, x) \otimes \bigvee_{N'\in\mathbb{N}} \bigwedge_{m,p\geq N'} a(x_m, x_p) \\
&= \bigvee_{N'\in\mathbb{N}} [\bigwedge_{N\in\mathbb{N}} \bigvee_{n\geq N} a(x_n, x) \otimes \bigwedge_{m,p\geq N'} a(x_m, x_p)] \\
&\leq \bigvee_{N'\in\mathbb{N}} [\bigvee_{n\geq N'} a(x_n, x) \otimes \bigwedge_{m,p\geq N'} a(x_m, x_p)] \\
&\leq \bigvee_{N'\in\mathbb{N}} \bigvee_{n\geq N'} [a(x_n, x) \otimes \bigwedge_{m\geq N'} a(x_m, x_n)] \\
&\leq \bigvee_{N'\in\mathbb{N}} \bigvee_{n\geq N'} \bigwedge_{m\geq N'} [a(x_n, x) \otimes a(x_m, x_n)] \\
&\leq \bigvee_{N'\in\mathbb{N}} \bigwedge_{m\geq N'} a(x_m, x).
\end{aligned}$$

□

Corollary 3.13. *If a sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, a) and $x \in X$, then:*

$$\bigvee_{N} \bigwedge_{n\geq N} a(x_n, x) = \bigwedge_{N} \bigvee_{n\geq N} a(x_n, x) \quad \text{and} \quad \bigvee_{N} \bigwedge_{n\geq N} a(x, x_n) = \bigwedge_{N} \bigvee_{n\geq N} a(x, x_n)$$

Proof. Since, by hypothesis, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then, by Lemma 3.12,

$$\begin{aligned}
\bigvee_{N} \bigwedge_{n\geq N} a(x_n, x) &\geq \bigwedge_{N} \bigvee_{n\geq N} a(x_n, x) \otimes k \\
&= \bigwedge_{N} \bigvee_{n\geq N} a(x_n, x),
\end{aligned}$$

for all $x \in X$. On the other hand, for any $N, N' \in \mathbb{N}$ and $M = \max\{N, N'\}$, then

$$\bigwedge_{n\geq N} a(x_n, x) \leq \bigwedge_{n\geq M} a(x_n, x) \leq \bigvee_{n\geq M} a(x_n, x) \leq \bigvee_{n\geq N'} a(x_n, x).$$

The inequality above is valid for any N and N'. Hence:

$$\bigvee_{N} \bigwedge_{n \geq N} a(x_n, x) \leq \bigwedge_{N'} \bigvee_{n \geq N'} a(x_n, x).$$

The proof of the second statement is similar. $\square$

Note that the second part of the proof is valid for any sequence in X.

Lemma 3.14. *Every* V*-functor sends Cauchy sequences to Cauchy sequences.*

Proof. For a V-functor $f : X \to Y$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in X, one has

$$\mathsf{Cauchy}_X(s) \leq \mathsf{Cauchy}_Y(f(s)),$$

where $f(s)$ denotes the sequence $(f(x_n))_{n\in\mathbb{N}}$ in Y. $\square$

As a consequence, if $f : X \to Y$ is a fully faithful V-functor then

$$\mathsf{Cauchy}_X(s) = \mathsf{Cauchy}_Y(f(s)).$$

Lemma 3.15. *If s' is a subsequence of a sequence s in a* V*-category X, then*

$$\mathsf{Cauchy}(s) \leq \mathsf{Cauchy}(s').$$

In particular, every subsequence of a Cauchy sequence is Cauchy.

Proof. Let $s = (x_n)_{n\in\mathbb{N}}$ be a sequence in $X = (X, a)$, $M \subseteq \mathbb{N}$ be an infinite subset of $\mathbb{N}$ and $s' = (x_m)_{m\in M}$. Then

$$\begin{aligned}
\mathsf{Cauchy}(s') &= \bigvee_{N\in M} \bigwedge_{\substack{n,m\geq N,\\ n,m\in M}} a(x_n, x_m) \\
&\geq \bigvee_{N\in M} \bigwedge_{n,m\geq N} a(x_n, x_m) \\
&= \bigvee_{N\in\mathbb{N}} \bigwedge_{n,m\geq N} a(x_n, x_m) \\
&= \mathsf{Cauchy}(s).
\end{aligned}$$ $\square$

To any sequence $s = (x_n)_{n\in\mathbb{N}}$ in a V-category (X, a) we associate a pair of V-distributors $\varphi_s : 1 \,{-\!\!\circ\!\!\to}\, X$ and $\psi_s : X \,{-\!\!\circ\!\!\to}\, 1$ defined as:

$$\varphi_s(x) = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} a(x_n, x) \text{ and } \psi_s(x) = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} a(x, x_n)$$

for all $x \in X$. One easily verifies that φ_s and ψ_s are V-distributors:

$$\begin{aligned}
\varphi_s(x) \otimes a(x,y) &= (\bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N} a(x_k,x)) \otimes a(x,y) \\
&= \bigvee_{N\in\mathbb{N}}((\bigwedge_{k\geq N} a(x_k,x)) \otimes a(x,y)) \\
&\leq \bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N}(a(x_k,x) \otimes a(x,y)) \\
&\leq \bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N} a(x_k,y) \\
&= \varphi_s(y),
\end{aligned}$$

and, similarly, $a(x,y) \otimes \psi_s(y) \leq \psi_s(x)$, for all $x, y \in X$.

Lemma 3.16. *For any sequence $s = (x_n)_{n\in\mathbb{N}}$ in a V-category (X,a):*

1. *For all $x, y \in X$, $\psi_s(x) \otimes \varphi_s(y) \leq a(x,y)$;*

2. $\mathsf{Cauchy}(s) \otimes \mathsf{Cauchy}(s) \leq \bigvee_{x\in X} \varphi_s(x) \otimes \psi_s(x) \leq \mathsf{Cauchy}(s)$.

Proof. Let $s = (x_n)_{n\in\mathbb{N}}$ be a sequence in a V-category (X,a). For all $x, y \in X$,

$$\begin{aligned}
\psi_s(x) \otimes \varphi_s(y) &= \bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N} a(x,x_n) \otimes \bigvee_{N'\in\mathbb{N}}\bigwedge_{n'\geq N'} a(x_{n'},y) \\
&= \bigvee_{N\in\mathbb{N}}(\bigwedge_{n\geq N} a(x,x_n) \otimes \bigwedge_{n'\geq N} a(x_{n'},y)) \\
&\leq \bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N}(a(x,x_n) \otimes a(x_n,y)) \\
&\leq a(x,y).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\bigvee_{x\in X} \varphi_s(x) \otimes \psi_s(x) &= \bigvee_{x\in X}(\bigvee_{N\in\mathbb{N}}\bigwedge_{m\geq N} a(x_m,x) \otimes \bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N} a(x,x_n)) \\
&\geq \bigvee_{N\in\mathbb{N}}\bigwedge_{m\geq N} a(x_m,x_N) \otimes \bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N} a(x_N,x_n) \\
&\geq \mathsf{Cauchy}(s) \otimes \mathsf{Cauchy}(s),
\end{aligned}$$

and, for any $x \in X$,

$$\begin{aligned}
\varphi_s(x) \otimes \psi_s(x) &= \Big(\bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} a(x_n,x)\Big) \otimes \Big(\bigvee_{M\in\mathbb{N}} \bigwedge_{m\geq M} a(x,x_m)\Big)\\
&= \bigvee_{N\in\mathbb{N}} \Big(\bigwedge_{n\geq N} a(x_n,x) \otimes \bigwedge_{m\geq N} a(x,x_m)\Big)\\
&\leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n,m\geq N} a(x_n,x) \otimes a(x,x_m)\\
&\leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n,m\geq N} a(x_n,x_m).
\end{aligned}$$

Hence $\bigvee_{x\in X} \varphi_s(x) \otimes \psi_s(x) \leq \mathsf{Cauchy}(s)$. $\square$

Theorem 3.17. *A sequence $s = (x_n)_{n\in\mathbb{N}}$ in (X,a) is Cauchy if and only if $\varphi_s \dashv \psi_s$ in* V-Dist.

Proof. If $s = (x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,a), then by Lemma 3.16 $\varphi_s \dashv \psi_s$. To prove the reverse, it is enough to consider (2) of Lemma 3.16. $\square$

Corollary 3.18. *Let s be a Cauchy sequence in a V-category X and s' be a subsequence of s. Then $\varphi_s = \varphi_{s'}$ and $\psi_s = \psi_{s'}$, where φ_s and ψ_s are the V-distributors induced by s and $\varphi_{s'}$ and $\psi_{s'}$ are the V-distributors induced by s'.*

Proof. By Lemma 3.15, s' is also Cauchy and therefore $\varphi_{s'} \dashv \psi_{s'}$. An easy calculation shows that $\varphi_s \leq \varphi_{s'}$ and $\psi_s \leq \psi_{s'}$, and the assertion follows from Lemma 2.17. $\square$

We have seen that every Cauchy sequence induces an adjunction of V-distributors. Naturally, the next problem that arises is if the reverse is also true, that is, if any adjunction is induced by a Cauchy sequence. The next theorem answers this question as it establishes the sufficient conditions which make this result valid.

Theorem 3.19. *Let V be a quantale with $k = \top$. If there is a sequence $(u_n)_{n\in\mathbb{N}}$ in V that satisfies*

i) $\bigvee_{n\in\mathbb{N}} u_n = k$,

ii) for all $n \in \mathbb{N}$, $u_n \ll k$,

iii) for all $n \in \mathbb{N}$, $u_n \leq u_{n+1}$,

then any adjunction $\varphi \dashv \psi : X \,{-\!\!\circ\!\!\rightarrow}\, 1$ of V-distributors is induced by a Cauchy sequence s in (X,a) such that $\varphi = \varphi_s$ and $\psi = \psi_s$.

Proof. We set up a sequence $(x_n)_{n\in\mathbb{N}}$ in X, considering, for each $n \in \mathbb{N}$, x_n such that

$$u_n \leq \varphi(x_n) \otimes \psi(x_n).$$

Since $k = \top$, for each $n \in \mathbb{N}$,

$$u_n \leq \varphi(x_n) \qquad \text{and} \qquad u_n \leq \psi(x_n).$$

Such a sequence is Cauchy because

$$\begin{aligned}\bigvee_{N\in\mathbb{N}} \bigwedge_{n,m\geq N} a(x_n, x_m) &\geq \bigvee_{N\in\mathbb{N}} \bigwedge_{n,m\geq N} \psi(x_n) \otimes \varphi(x_m) \\ &\geq \bigvee_{N\in\mathbb{N}} \bigwedge_{n,m\geq N} u_n \otimes u_m \\ &\geq \bigvee_{N\in\mathbb{N}} u_N \otimes u_N \\ &\geq k,\end{aligned}$$

where the last inequality follows from the fact that $(u_n)_{n\in\mathbb{N}}$ satisfies i and iii.
Moreover, for every $x \in X$ and $n \in \mathbb{N}$,

$$u_n \otimes a(x_n, x) \leq \varphi(x_n) \otimes a(x_n, x) \leq \varphi(x).$$

Therefore,

$$a(x_n, x) \leq \mathsf{hom}(u_n, \varphi(x)).$$

Hence, for all $x \in X$ and all $N \in \mathbb{N}$,

$$\begin{aligned}\bigwedge_{n\geq N} a(x_n, x) &\leq \bigwedge_{n\geq N} \mathsf{hom}(u_n, \varphi(x)) \\ &= \mathsf{hom}(\bigvee_{n\geq N} u_n, \varphi(x)) \\ &= \mathsf{hom}(k, \varphi(x)) \\ &= \varphi(x),\end{aligned}$$

and, consequently, $\varphi_s \leq \varphi$. Similarly, $\psi_s \leq \psi$, and Lemma 2.17 implies $\varphi_s = \varphi$ and $\psi_s = \psi$. □

Corollary 3.20. *Under the conditions of Theorem 3.19, if (X, a) is a symmetric* V-*category and $\varphi \dashv \psi : X {\longrightarrow\!\!\!\!\circ}\, 1$ then, for any $x \in X$, $\varphi(x) = \psi(x)$.*

Proof. If (X, a) is a symmetric V-category and $\varphi \dashv \psi : X {\longrightarrow\!\!\!\!\circ}\, 1$ then, by Theorem 3.19, there is a sequence $s = (x_n)_{n\in\mathbb{N}}$ in X such that $\varphi = \varphi_s$ and $\psi = \psi_s$. Thus, for any $x \in X$,

$$\varphi(x) = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} a(x_n, x) = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} a(x, x_n) = \psi(x). \qquad \square$$

Examples 3.21.

1. Any sequence of elements of $\mathsf{V} = 2$ containing the element true fulfills the conditions of Theorem 3.19.

2. The sequence $u_n = \frac{1}{n}$ in the quantale $[0, +\infty]$ is such that, for all $n \in \mathbb{N}$, $u_n > 0$, $u_n > u_{n+1}$, and, furthermore $0 = \inf_{n\in\mathbb{N}} u_n$.

3. In the quantal Δ the top element is the neutral element k and the sequence $(f_{\frac{1}{n},1-\frac{1}{n}})_{n\in\mathbb{N}}$ satisfies the conditions of Theorem 3.19:

$$\bigvee_{n\in\mathbb{N}} f_{\frac{1}{n},1-\frac{1}{n}} = k, \qquad \forall n \in \mathbb{N}, f_{\frac{1}{n},1-\frac{1}{n}} \ll k, \qquad \forall n \in \mathbb{N}, f_{\frac{1}{n},1-\frac{1}{n}} \leq f_{\frac{1}{n+1},1-\frac{1}{n+1}}.$$

Lemma 3.22. *Let $f : X \to Y$ be a V-functor, $s = (x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X with associated adjunction $\varphi_s \dashv \psi_s$ in V-Dist. Then $\varphi_{f(s)} = f_* \cdot \varphi_s$ and $\psi_{f(s)} = \psi_s \cdot f^*$, where $f(s)$ denotes the sequence $(f(x_n))_{n\in\mathbb{N}}$ in Y.*

Proof. By Lemma 2.17, it is enough to show that $\varphi_{f(s)} \geq f_* \cdot \varphi_s$ and $\psi_{f(s)} \geq \psi_s \cdot f^*$. In fact, for any $y \in Y$,

$$\begin{aligned}
f_* \cdot \varphi_s(y) &= \bigvee_{x\in X} \varphi_s(x) \otimes b(f(x), y) \\
&= \bigvee_{x\in X} \left(\bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} a(x_n, x) \right) \otimes b(f(x), y) \\
&\leq \bigvee_{x\in X} \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} b(f(x_n), f(x)) \otimes b(f(x), y) \\
&\leq \varphi_{f(s)}(y),
\end{aligned}$$

and the inequality $\psi_{f(s)} \geq \psi_s \cdot f^*$ follows similarly. $\square$

To finish this section, we will see that functors induced by lax morphisms of quantales preserve Cauchy sequences.

Lemma 3.23. *Let $F : \mathsf{V} \to \mathsf{W}$ be a lax morphism of quantales and (X, a) be a V-category. If $s = (x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, a) then s is a Cauchy sequence in $(X, F \cdot a)$.*

Proof. If $s = (x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, a) then

$$\begin{aligned} F(k) &\leq F(\bigvee_{N\in\mathbb{N}} \bigwedge_{n,m\geq N} a(x_m, x_n)) \\ \Rightarrow k &\leq \bigvee_{N\in\mathbb{N}} F(\bigwedge_{n,m\geq N} a(x_m, x_n)) \\ \Rightarrow k &\leq \bigvee_{N\in\mathbb{N}} \bigwedge_{n,m\geq N} F \cdot a(x_m, x_n). \end{aligned}$$

Hence s is a Cauchy sequence in the W-category $(X, F \cdot a)$. □

3.3 Convergence in a V-category

In the last section it was established that, in a V-category (X, a), any sequence induces a pair of V-distributors which determine one another. Even more important was setting the conditions that ensure that a sequence is Cauchy precisely if the induced V-distributors form an adjunction. The next goal is, of course, to establish the relationship between convergence and representability by establishing the range of conditions that allow us to have equivalence between the two concepts.

Definition 3.24. A sequence $s = (x_n)_{n\in\mathbb{N}}$ in a V-category (X, a) converges to $x \in X$ if x is in the closure of every subsequence of s, that is,

$$s \longrightarrow x \quad \text{whenever} \quad x \in \overline{\{x_m, m \in M\}}, \text{ for any infinite subset } M \text{ of } \mathbb{N},$$

or, equivalently,

$$s \longrightarrow x \quad \text{whenever} \quad k \leq \bigvee_{m\in M} a(x, x_m) \otimes a(x_m, x), \text{ for any infinite subset } M \text{ of } \mathbb{N}.$$

Examples 3.25.

1. For $\mathsf{V} = 2$, a sequence $s = (x_n)_{n\in\mathbb{N}}$ in an ordered set X converges to x if, for any $M \subseteq \mathbb{N}$ infinite, there is $m \in M$ such that $x \leq x_m \leq x$. Since this is valid for all infinite subsets of $\mathbb{N}$, if the order relation in X is anti-symmetric, then we can conclude that s converges to x if there is $m \in \mathbb{N}$ such that $x_n = x$ for all $n \geq m$.

2. A sequence $s = (x_n)_{n\in\mathbb{N}}$ in a $[0, +\infty]$-category (X, a) converges to $x \in X$ if, for any $M \subseteq \mathbb{N}$ infinite,
$$0 = \inf_{m\in M} a(x, x_m) + a(x_m, x),$$

or, equivalently, if, for any $M \subseteq \mathbb{N}$ infinite,

$$\begin{aligned}&\forall \epsilon > 0, \exists m \in M : \epsilon \geq a(x, x_m) + a(x_m, x)\\ \Leftrightarrow &\forall \epsilon > 0, \exists m \in M : \epsilon \geq a(x, x_m) \text{ and } \epsilon \geq a(x_m, x).\end{aligned}$$

Since this is valid for any infinite subset of $\mathbb{N}$, we can state that s converges to x if

$$\forall \epsilon > 0, \exists m \in M : \forall n \geq m \epsilon \geq a(x, x_n) \text{ and } \epsilon \geq a(x_n, x),$$

which is the usual definition of convergence of a sequence in metric spaces.

3. Finally, a sequence $s = (x_n)_{n\in\mathbb{N}}$ in a probabilistic metric space (X, a) converges to $x \in X$ if, for any $M \subseteq \mathbb{N}$ infinite,

$$k = \bigvee_{m\in M} a(x, x_m) \otimes a(x_m, x) \Leftrightarrow \qquad 1 = \bigvee_{m\in M} a(x, x_m) \otimes a(x_m, x)(t)$$
$$\Leftrightarrow \forall \epsilon < 1, \exists m \in M : \epsilon \leq a(x, x_m) \otimes a(x_m, x)(t).$$

for all $t > 0$.

Lemma 3.26. *If a sequence $s = (x_n)_{n\in\mathbb{N}}$ converges to x in a V-category (X, a) and $f : X \to Y$ is a V-functor then $f(s)$ converges to $f(x)$ in the V-category (Y, b), where $f(s)$ denotes the sequence $(f(x_n))_{n\in\mathbb{N}}$ in Y.*

Proof. In Subsection 3.1 it was proved that V-functors are continuous maps. Thus, they preserve convergence. □

The following results relate the convergence of a Cauchy sequence in a V-category with the representability of the adjunction induced by such sequence.

Theorem 3.27. *Let V be a quantale and $s = (x_n)_{n\in\mathbb{N}}$ a Cauchy sequence in a V-category (X, a). If $\varphi_s = x_*$ and $\psi_s = x^*$ then s converges to $x \in X$.*

Proof. One must show that $x \in \overline{\{x_m, M \in \mathbb{N}\}}$, for every $M \subseteq \mathbb{N}$ infinite. Let M be an infinite subset of $\mathbb{N}$. By Corollary 3.18, any subsequence $s' = (x_m)_{m\in M}$ of s induces the same adjunction as s; then, for $A = \{x_m : m \in M\}$,

$$\begin{aligned} & k \leq \bigvee_{y \in A} \varphi_s(y) \otimes \psi_s(y) \\ \Leftrightarrow & k \leq \bigvee_{m \in M} \varphi_s(x_m) \otimes \psi_s(x_m) \\ \Leftrightarrow & k \leq \bigvee_{m \in M} a(x, x_m) \otimes a(x_m, x). \end{aligned}$$

Thus s converges to x. $\square$

Hence the representability of the adjunction $\varphi_s \dashv \psi_s$ guarantees the convergence of the Cauchy sequence s. The reciprocal is also true if the neutral element of V coincides with the top element. In order to prove this, we will the following additional result.

Proposition 3.28. *Let $s = (x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in a V-category X and consider the sequence $\widetilde{s} := (x_n^*)_{n \in \mathbb{N}}$ in $\widetilde{X}$. Then $\widetilde{s} \to \psi_s$ in $\hat{X}$. Moreover, if $k = \top$ in V, then $\widetilde{s} \to \psi$ implies $\psi = \psi_s$, for every V-distributor $\psi \in \hat{X}$.*

Proof. Since $\psi_s^* = \psi_s \cdot \mathcal{Y}_X^*$ by Lemma 2.21, and $\psi_s \cdot \mathcal{Y}_X^* = \psi_{\widetilde{s}}$ by Lemma 3.22, then $\psi_s^* = \psi_{\widetilde{s}}$. Hence, by Theorem 3.27, $\widetilde{s} \to \psi_s$.
Let $k = \top$ in V and suppose that $\widetilde{s} \to \psi$. Since $\psi \in \overline{\mathcal{Y}_X(X)}$ the V-distributor $\psi : X \multimapdotinv 1$ has a left adjoint $\varphi : 1 \multimapdotinv X$. Then, for any $M \subseteq \mathbb{N}$ infinite,

$$\begin{aligned} k &\leq \bigvee_{m \in M} \hat{a}(\psi, x_m^*) \otimes \hat{a}(x_m^*, \psi) \\ &= \bigvee_{m \in M} \varphi(x_m) \otimes \psi(x_m) \\ &\leq \bigvee_{n \in \mathbb{N}} \varphi(x_n) \otimes \psi(x_n), \end{aligned}$$

and, for any $n \in \mathbb{N}$ and $x \in X$,

$$a(x, x_n) \otimes \psi(x_n) \otimes \varphi(x_n) \leq \psi(x) \otimes \varphi(x_n) \leq \psi(x).$$

Therefore, for any $N \in \mathbb{N}$ and $x \in X$,

$$\begin{aligned} \psi(x) &\geq \mathsf{hom}(\bigvee_{n \geq N} \varphi(x_n) \otimes \psi(x_n), \psi(x)) \\ &= \bigwedge_{n \geq N} \mathsf{hom}(\varphi(x_n) \otimes \psi(x_n), \psi(x)) \\ &\geq \bigwedge_{n \geq N} a(x, x_n). \end{aligned}$$

Thus $\psi \geq \psi_s$ and similarly one obtains $\varphi \geq \varphi_s$ and Lemma 2.17 guarantees that $\psi = \psi_s$. □

Corollary 3.29. *Let* V *be a quantale with* $k = \top$ *and* $s = (x_n)_{n\in\mathbb{N}}$ *a Cauchy sequence in a* V*-category* (X, a)*. If* s *converges to* $x \in X$ *then* $\varphi_s = x_*$ *and* $\psi_s = x^*$*.*

Proof. From $s \to x$ it follows that $\widetilde{s} \to x^*$ in $\hat{X}$, where $\widetilde{s} := (x_n^*)_{n\in\mathbb{N}}$ and $s = (x_n)_{n\in\mathbb{N}}$, and therefore $x^* = \psi_s$ by the proposition above. □

Corollary 3.30. *Let* V *be a quantale with* $k = \top$ *and* $s = (x_n)_{n\in\mathbb{N}}$ *a Cauchy sequence in a* V*-category* (X, a)*. If* s *converges to* $x \in X$ *then* $\varphi_s(x) = k$ *and* $\psi(x) = k$*.*

As in the last section, we end this section by analysing the behaviour of functors induced by morphisms of quantales on the convergence of sequences.

Lemma 3.31. *Let* $F : \mathsf{V} \to \mathsf{W}$ *be a morphism of quantales. If* $s = (x_n)_{n\in\mathbb{N}}$ *is a sequence in a* V*-category* (X, a) *convergent to* $x \in X$ *then* $s = (x_n)_{n\in\mathbb{N}}$ *also converges to* x *in the* W*-category* $(X, F \cdot a)$*.*

Proof. Since F is monotone, then, for all $M \subseteq \mathbb{N}$ infinite,

$$\begin{aligned} F(k) &\leq F(\bigvee_{m\in M} a(x, x_m) \otimes a(x_m, x)) \\ \Leftrightarrow k &\leq \bigvee_{m\in M} F \cdot a(x, x_m) \otimes F \cdot a(x_m, x). \end{aligned}$$

Then $s = (x_n)_{n\in\mathbb{N}}$ also converges to $x \in X$ in the W-category $(X, F \cdot a)$. □

Therefore functors induced by morphisms of quantales preserve convergence.

3.4 L-Complete V-categories

Having established the conditions under which the convergence of a Cauchy sequence is equivalent to the representability of the induced adjunction, one can now focus on the notion of completeness.

Definition 3.32. A V-category X is *L-complete* if every left adjoint V-distributor $\varphi : Y \,{\circ\!\!\!\rightarrow}\, X$ is of the form $\varphi = f_*$ for some V-functor $f : Y \to X$ (or, equivalently, if every right adjoint V-distributor $\psi : X \,{\circ\!\!\!\rightarrow}\, Y$ is of the form $\psi = f^*$ for some V-functor $f : Y \to X$).

Since it is enough to consider the case $Y = 1$, where $1 = (1, k)$, a V-category is L-complete if and only if every left adjoint V-distributor $\varphi : 1 \,{\circ\!\!\!\rightarrow}\, X$ is of the form

$\varphi = x_*$, for some $x \in X$. Or, in other words, a V-category is L-complete if and only if every pair of adjoint V-distributors, $(\varphi : 1 \multimap X) \dashv (\psi : X \multimap 1)$, is representable in the sense that there is $x \in X$ such that $\varphi = x_*$ and $\psi = x^*$. As a consequence:

Lemma 3.33. *The following assertions are equivalent:*

i) X *is L-complete;*

ii) $(-)_* : X \to \{\varphi : 1 \multimap X : \ \varphi$ *is left adjoint*$\}$ *is surjective;*

iii) $(-)^* : X \to \{\psi : X \multimap 1 : \ \psi$ *is rigth adjoint*$\}$ *is surjective.*

It was seen that, under the conditions of Theorem 3.19, the convergence of Cauchy sequences is equivalent to the representability of the induced adjoint distributors. This means that

Theorem 3.34. *Under the conditions of Theorem 3.19, a* V*-category* X *is L-complete if and only if it is Cauchy complete*

There are some important results relating L-completeness and closed subsets of a V-category.

Proposition 3.35. *1. Every closed subset of a L-complete* V*-category is L-complete;*

2. Every L-complete V*-subcategory of a separated* V*-category is closed.*

Furthermore, if A *is a subset of a* V*-category* X*, the inclusion* V*-functor* $i : A \to X$ *is fully dense if and only if* $\overline{A} = X$*.*

Proof. 1. Let A be a closed subset of a L-complete V-category X. The composite $m_* \cdot \varphi$, where $\varphi : 1 \multimap A$ is a left adjoint V-distributor and m is the embedding $m : A \to X$, is a left adjoint V-distributor. Since X is L-complete, there is $x \in X$ such that $x_* = m_* \cdot \varphi$ and, by Proposition 3.2 (vi), $x \in \overline{A} = A$. Thus $\varphi = x_*$, due to the fully faithfulness of m.

2. Let A be a L-complete V-subcategory of a separated V-category X and $x \in \overline{A}$. By Proposition 3.2 (v), $m^* \cdot x_* \dashv x^* \cdot m_*$ and, since A is L-complete, there is $y \in A$ such that $y_* = m^* \cdot x_*$ and $y^* = x^* \cdot m_*$. Hence, $m(y)_* = m_* \cdot y_* = m_* \cdot m^* \cdot x_* \leq x_*$ and $m(y)^* = y^* \cdot m^* = x^* \cdot m_* \cdot m^* \geq x^*$, which, by Lemma 2.17 and by the fact that X is separated, implies $x = y$. Thus $x \in A$.

Finally, consider a V-category X, $A \subseteq X$ and the inclusion V-functor $i : A \to X$. If $i_* \cdot i^* = a$ and $x \in X$, then

$$\begin{aligned} &\bigvee_{y \in A} i^*(x,y) \otimes i_*(y,x) = a(x,x) \\ \Leftrightarrow &\bigvee_{y \in A} a(x,y) \otimes a(y,x) = a(x,x). \end{aligned}$$

Thus $x \in \overline{A}$.
To prove the reverse, suppose that $\overline{A} = X$; then, by 3.2 (ii), for all $x \in \overline{A} = X$,

$$a(x,x) \leq \bigvee_{y \in A} a(x,y) \otimes a(y,x) = \bigvee_{y \in A} a(x,i(y)) \otimes a(i(y),x),$$

which means that i is fully dense. $\square$

For any V-category X, $\hat{X}$ is L-complete since a right adjoint V-distributor $\psi : \hat{X} {\longrightarrow\!\!\!\!\!\circ}\, 1$ is represented by the V-functor $\psi \cdot (\mathcal{Y}_X)_* \in \hat{X}$. Thus, by Lemma 3.7 and Proposition 3.35, $\widetilde{X}$ is L-complete and $\mathcal{Y}_X : X \to \widetilde{X}$ is fully dense (and fully faithful). Hence $\mathcal{Y}_X : X \to \widetilde{X}$ provides a L-completion of X.
The fact that $\mathcal{Y}_X : X \to \widetilde{X}$ is fully faithful and fully dense implies that the V-distributor $(\mathcal{Y}_X)_*$ is an isomorphism with inverse $(\mathcal{Y}_X)^*$. Thus, if X and Y are V-categories and Y is separated and L-complete, for every V-functor $f : X \to Y$ there is a unique V-functor $g : \widetilde{X} \to Y$ such that $g \cdot \mathcal{Y}_X = f$. In fact, g is the V-functor that represents the left adjoint V-distributor $f_* \cdot (\mathcal{Y}_X)^*$. Since Y is L-complete and separated, g exists and is unique. Thus,

Theorem 3.36. *The full subcategory of separated and L-complete* V*-categories is reflective in* V-Cat. *The reflection map for a* V*-category X can be chosen as $\mathcal{Y}_X : X \to \widetilde{X}$.*

We can also characterise L-complete V-categories through the functor $\mathcal{Y}_X : X \to \widetilde{X}$.

Theorem 3.37. *The following statements are equivalent:*

i) X is L-complete;

ii) $\mathcal{Y}_X : X \to \widetilde{X}$ has a left adjoint;

iii) $\mathcal{Y}_X : X \to \widetilde{X}$ is invertible.

Proof. If X is a L-complete V-category, every element of $\widetilde{X}$ is representable by some $x \in X$, which means that there is a map $S_X : \widetilde{X} \to X$, that associates to each right adjoint V-distributor $\psi : X \longrightarrow\!\!\!\!\!\!\circ\;\; 1$ some element x in X such that $\psi = x^*$. For any $\psi \in \widetilde{X}$,

$$\mathcal{Y}_X \cdot S_X(\psi) = \psi,$$

and, for any $x \in X$,

$$S_X \cdot \mathcal{Y}_X(x) = y \quad \text{ with } \quad x^* = y^*,$$

for some $y \in Y$. But $x^* = y^*$ implies that,

$$k \le a(y, x) = a(S_X \cdot \mathcal{Y}_X(x), x),$$

which is equivalent to $S_X \cdot \mathcal{Y}_X(x) \le x$.
To prove that (ii) implies (iii), suppose that $\mathcal{Y}_X : X \to \widetilde{X}$ has a left adjoint S_X. Since $\mathcal{Y}_X$ is fully faithful then $S_X \cdot \mathcal{Y}_X(x) \simeq 1_X$. The fact that $\mathcal{Y}_X \cdot S_X \cdot \mathcal{Y}_X = \mathcal{Y}_X = 1_{\widetilde{X}} \cdot \mathcal{Y}_X$ and the fully density of $\mathcal{Y}_X$ guarantee that $\mathcal{Y}_X \cdot S_X = 1_{\widetilde{X}}$.
Finally, the existence of S_X inverse and, consequently, left adjoint to $\mathcal{Y}_X$ guarantees the representability of any element of $\widetilde{X}$. □

Lemma 3.38. *Under the conditions of Theorem 3.19, if a* V*-subcategory* A *of* X *is symmetric then also* $\overline{A}$ *is symmetric. In particular, the L-completion of a symmetric* V*-category is also symmetric.*

Proof. Let A be a symmetric V-subcategory of X; by Corollary 3.20 we have that $\varphi(x) = \psi(x)$ for any adjunction $\varphi \dashv \psi : A \longrightarrow\!\!\!\!\!\!\circ\;\; 1$ and any $x \in A$. In particular, for any $x \in A$, $a(x, -) = a(-, x)$. If $y \in \overline{A}$ then $a(x, y) = a(y, x)$; if also $y' \in A$:

$$a(y, y') = \bigvee_{x \in A} a(y, x) \otimes a(x, y') = \bigvee_{x \in A} a(x, y) \otimes a(y', x) = a(y', y). \qquad \square$$

A general investigation about the relationship between symmetrisation and L-completion of quantaloids enriched categories can be found in [HS11].

Definition 3.39. A V-category X is *L-injective* if it is injective with respect to fully faithful and fully dense V-functors, that is if, for every fully faithful and fully dense V-functor $i : A \to B$ and every V-functor $f : A \to X$, there is a V-functor $g : B \to X$ such that $g \cdot i \simeq f$.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & B \arrow[d, dashed, "g"] \\ & X \end{tikzcd}$$

Recall that a V-category X is *injective* if for all V-functors $f : A \to X$ and $i : A \to B$, with i fully faithful, there is a V-functor $g : B \to X$ such that $g \cdot i = f$. Therefore every injective V-category is L-injective.

Proposition 3.40. *A* V*-category* X *is L-complete if and only if* X *is L-injective.*

Proof. If X is L-complete the Yoneda functor, $\mathcal{Y}_X$, has a left adjoint, S_X. Given a V-functor $f : Y \to X$ and a fully faithful and fully dense V-functor $i : Y \to Z$, consider the V-functor g given by the composite of V-functors $S_X \cdot [(-) \cdot f^*] \cdot [(-) \cdot i_*] \cdot \mathcal{Y}_Z$ as shown in the diagram below.

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[dr, "f"'] & Z \arrow[r, "\mathcal{Y}_Z"] \arrow[d, dashed, "g"] & \widetilde{Z} \arrow[r, "(-)\cdot i_*"] & \widetilde{Y} \arrow[d, "(-)\cdot f^*"] \\
& X & & \widetilde{X} \arrow[ll, "S_X"]
\end{tikzcd}$$

For any $y \in Y$,

$$\begin{aligned}
g \cdot i(y) &= S_X \cdot [(-) \cdot f^*] \cdot [(-) \cdot i_*] \cdot \mathcal{Y}_Z \cdot i(y) \\
&= S_X \cdot [(-) \cdot f^*] \cdot [(-) \cdot i_*](y^* \cdot i^*),
\end{aligned}$$

and, due to fully faithfulness of i,

$$g \cdot i(y) = S_X \cdot [(-) \cdot f^*](y^*) = S_X(f(y)^*) \simeq f(y).$$

Thus, X is L-injective.
If (X, a) is L-injective, there is $g : \widetilde{X} \to X$ such that $g \cdot \mathcal{Y}_X = 1_X$. Thus $\mathcal{Y}_X \cdot g \cdot \mathcal{Y}_X = \mathcal{Y}_X$ and, since $\mathcal{Y}_X$ is fully dense, we conclude that g is left adjoint to $\mathcal{Y}_X$. Hence, X is L-complete.

$$\begin{tikzcd}
X \arrow[r, "\mathcal{Y}_X"] \arrow[dr, "1_X"'] & \widetilde{X} \arrow[d, dashed, "g"] \\
& X
\end{tikzcd}$$

□

Lemma 3.41. *For any quantale* V*, the category* $(\mathsf{V}, \mathsf{hom})$ *is injective.*

Proof. Any V-functor $f : Y \to \mathsf{V}$ can be seen as a V-distributor $f : 1 {\longrightarrow\!\!\!\!\!\circ\,} Y$ and given a fully faithful V-functor $i : Y \to Z$, the V-functor $g = i_* \cdot f$ is such that $g \cdot i = f$. □

Therefore, the V-category $(\mathsf{V}, \mathsf{hom})$ is L-injective and, therefore, L-complete.

Examples 3.42. 1. An adjunction $\varphi \dashv \psi : X {\longrightarrow\!\!\!\!\!\circ\,} 1$ of 2-distributors identifies an $x \in X$ such that $\varphi(x)$ and $\psi(x)$ and, for any $y, y' \in X$, if $\psi(y)$ and $\varphi(y')$ then $y \leq y'$. Therefore $\varphi = x_* = \uparrow x$ and $\psi = x^* = \downarrow x$. In conclusion, any adjunction of 2-distributors is representable meaning that any ordered set is always L-complete.

2. A metric space X is L-complete if every adjunction $\varphi \dashv \psi : X \relbar\!\!\!\circ\!\!\!\rightarrow 1$ of $[0,+\infty]$-distributors is representable, or, equivalently, if and only if every Cauchy sequence converges. An L-completion of X is given by $\mathcal{Y}_X : X \to \widetilde{X}$, where $\widetilde{X}$ is the subset of $\hat{X}$ defined by all right adjoint $[0,+\infty]$-distributors of type $X \relbar\!\!\!\circ\!\!\!\rightarrow 1$ and, by Lemma 3.38, if X is symmetric $\widetilde{X}$ is also symmetric.

3. As for metric spaces, a probabilistic metric space X is L-complete if every left adjoint Δ-distributor $\varphi : 1 \relbar\!\!\!\circ\!\!\!\rightarrow X$ is representable, that is, if every Cauchy sequence converges, and a L-completion of X is given by $\mathcal{Y}_X : X \to \widetilde{X}$. By Lemma 3.38, the L-completion $\widetilde{X}$ of a symmetric probabilistic metric space $X = (X, a)$ is also symmetric. If $X = (X, a)$ is finitary, $\widetilde{X}$ is finitary as well. To see that $\widetilde{X}$ is finitary, we show first that $\psi(x) \in \Delta$ is finite, for every $x \in X$ and every right adjoint Δ-distributor $\psi : X \relbar\!\!\!\circ\!\!\!\rightarrow 1$. Let $x \in X$, and put $\delta = \psi(x)(\infty)$. Since $k = \bigvee_{x' \in X} \psi(x')$, for every $\epsilon < 1$ there is some $x' \in X$ with $f_{1,\epsilon} \leq \psi(x')$. Hence,

$$\begin{aligned} 1 &= a(x, x')(\infty) \\ &= \mathsf{hom}(\psi(x'), \psi(x))(\infty) \\ &\leq \mathsf{hom}(f_{1,\epsilon}, f_{0,\delta})(\infty) \\ &= f_{0,\delta \oslash \epsilon}(\infty) \\ &= \delta \oslash \epsilon \end{aligned}$$

for all $\epsilon < 1$, which implies $\delta = 1$. Given also $\psi' : X \relbar\!\!\!\circ\!\!\!\rightarrow 1$ in $\widetilde{X}$, the distance

$$\begin{aligned} \hat{a}(\psi, \psi') &= \bigvee_{x \in X} \hat{a}(\psi, x^*) \otimes \hat{a}(x^*, \psi') \\ &= \bigvee_{x \in X} \psi(x) \otimes \psi'(x) \end{aligned}$$

is finite.

The first construction of a L-completion of a probabilistic metric space was given by Sherwood [She66] by putting an appropriate probabilistic metric on the set of equivalence classes of Cauchy sequences of a given space. A study of probabilistic metric spaces as enriched categories can be found in [Cha09], where it is also shown that the categorical notion of L-completeness is equivalent to the traditional one based on Cauchy sequences.

3.5 Morphisms of quantales and L-completeness

In this section we will study the behavior of functors induced by morphisms of quantales with respect to L-complete V-categories. We have seen that a morphism of quantales $F : \mathsf{V} \to \mathsf{W}$ induces functors $F : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{W}\text{-}\mathsf{Cat}$ and $F : \mathsf{V}\text{-}\mathsf{Dist} \to \mathsf{W}\text{-}\mathsf{Dist}$, and that the latter one preserves adjunctions. Hence, the map

$$\Phi : \{\varphi : 1 \multimap X : \varphi \text{ is left adjoint}\} \to \{\varphi' : 1 = F1 \multimap FX : \varphi' \text{ is left adjoint}\},$$

that takes any left adjoint V-distributor $\varphi : 1 \multimap X$ to the left adjoint V-distributor $\Phi\varphi = F\varphi$, makes the following diagram commute:

$$\begin{tikzcd}
\{\varphi : 1 \multimap X : \varphi \text{ is left adjoint}\} \arrow[rr, "\Phi"] & & \{\varphi' : 1 = F1 \multimap FX : \varphi' \text{ is left adjoint}\} \\
& {|X| = |FX|} \arrow[ul, "(-)_*"] \arrow[ur, "(-)_*"'] &
\end{tikzcd}$$

where $|Y|$ denotes the underlying set of a V-category Y. Since X (respectively FX) is L-complete if and only if the map $(-)_*$ is surjective, we find

Proposition 3.43. *1. If FX is L-complete and Φ is injective, then X is L-complete.*

2. If X is L-complete and Φ is surjective, then FX is L-complete.

Proof. Suppose that FX is L-complete and Φ is injective and let $\varphi : 1 \multimap X$ be a left adjoint V-distributor. Since FX is L-complete there is $x \in |X| = |FX|$ such that $\Phi\varphi = x_*$ and $\Phi\varphi = \Phi x_*$. Hence, by the injectivity of Φ, $\varphi = x_*$. To prove the second statement, let X be a L-complete category and Φ surjective. Given $\varphi' : F1 \multimap FX$ there is $\varphi : 1 \multimap X$ such that $\Phi\varphi = \varphi'$ and $\varphi = x_*$, for some $x \in X$, because X is L-complete. Then $\varphi' = x_*$. $\square$

The following corollary translates the preceding proposition but now setting conditions on the morphism of quantales F in order to achieve the same results.

Corollary 3.44. *Let $F : \mathsf{V} \to \mathsf{W}$ and $G : \mathsf{W} \to \mathsf{V}$ be morphisms of quantales*

1. *If FX is L-complete and F is injective, then X is L-complete.*

2. *Assume that either $G \dashv F$ or that $F \dashv G$ and F is injective. If X is L-complete then FX is L-complete.*

Proof. Certainly, if $F : \mathsf{V} \to \mathsf{W}$ is injective, then Φ is injective for every V-category X. Suppose that $F \dashv G$ and F is injective and consider the adjunction of W-distributors

$(\varphi' : 1 \looparrowright FX) \dashv (\psi' : FX \looparrowright 1)$. Applying G gives the adjunction $(G\varphi' : 1 \looparrowright GFX) \dashv (G\psi' : GFX \looparrowright 1)$ in V-Dist. Since $GF = 1_{\mathsf{V}}$, $G\varphi'$ is of type $1 \looparrowright X$ and $G\psi'$ is of type $X \looparrowright 1$, then FG is of the type $\varphi' 1 \looparrowright FX$ and $FG\psi'$ is of type $FX \looparrowright 1$. Thus $FG\varphi' \leq \varphi'$ and $FG\psi' \leq \psi'$ and Lemma 2.17 implies $FG\varphi' = \varphi'$ Therefore Φ is surjective.
Suppose now that $G \dashv F$. Given an adjunction $(\varphi' : 1 \looparrowright FX) \dashv (\psi' : FX \looparrowright 1)$ in W-Dist, since the identity map on X can be seen as the V-functor $\gamma : GFX \to X$, the adjunction $(G\varphi' : 1 \looparrowright GFX) \dashv (G\psi' : GFX \looparrowright 1)$, in V-Dist, can be composed with $\gamma_* \dashv \gamma^*$ to yield $(\gamma_* \cdot G\varphi' : 1 \looparrowright X) \dashv (G\psi' \cdot \gamma^* : X \looparrowright 1)$. Furthermore, $F(\gamma_* \cdot G\varphi') = F \cdot G\varphi' \geq \varphi'$ and $F(G\psi' \cdot \gamma^*) = F \cdot G\psi' \geq \psi'$, since $F\gamma$ is the identity on FX due to $FGF = F$. Thus $\Phi(\gamma_* \cdot G\varphi') = \varphi'$ follows again from Lemma 2.17. □

The Proposition 3.40 provides an alternative way to prove preservation of L-completeness by functors.

Theorem 3.45. *Let $G : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{W}\text{-}\mathsf{Cat}$ be a locally monotone functor with left adjoint $F : \mathsf{W}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$. If F sends fully faithful and fully dense W-functors to fully faithful and fully dense V-functors, then G sends L-complete V-categories to L-complete W-categories.*

Proof. We write $\eta : 1 \to GF$ and $\varepsilon : FG \to 1$ for the units of the adjunction $F \dashv G$. Let X be a L-complete V-category, $i : Y \to Z$ be a fully faithful and fully dense W-functor and $f : Y \to GX$ be a W-functor. Since Fi is a fully faithful and fully dense V-functor, there is a V-functor $g : FZ \to X$ with $g \cdot Fi \simeq \varepsilon_X \cdot Ff$. Then

$$Gg \cdot \eta_Z \cdot i = Gg \cdot GFi \cdot \eta_Y \simeq G\varepsilon_X \cdot GFf \cdot \eta_Y = G\varepsilon_X \cdot \eta_{GX} \cdot f = f. \qquad \square$$

Lemma 3.46. *Let $F : \mathsf{W} \to \mathsf{V}$ be a morphism of quantales. Then the induced functor $F : \mathsf{W}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$ sends fully faithful W-functors to fully faithful V-functors, and sends fully dense W-functors to fully dense V-functors.*

Proof. This follows immediately from the commutativity of the diagrams (2.1) in Subsection 2.3. □

Example 3.47. In Example 1.16 it was seen that the functor $I_\infty : \mathsf{Met} \to \Delta$ induced by a morphism of quantales has a left adjoint O_∞. Then, by Proposition 3.43 and Corollary 3.44, a metric space X is L-complete if and only if the probabilistic metric space $I_\infty(X)$ is L-complete. Since also $I_\infty \dashv P_\infty$, by Theorem 3.45 and Lemma 3.46, P_∞ preserves L-completness, that is, $P_\infty X$ is a L-complete metric space whenever X is a L-complete probabilistic metric space.

3.6 Comparison with related work

In a completely distributive quantale V with $k = \top$, Flagg [Fla97] considers that, in a V-category (X, a), the open ball of center x and radius ϵ, with $\epsilon \ll k$ and $x \in X$, is the set of elements y in X such that $\epsilon \ll a(x, y)$, that $U \subseteq X$ is open if and only if, for any $x \in U$, there is an open ball centered at x and contained in U, and that a topology in (X, a) is the collection of open subsets of X.

Example 3.48. In a probabilistic metric space (X, a) the definition of open ball centered at $x \in X$, with radius $f \ll k$ in Δ, becomes:

$$B(f, x) = \{y \in X : f \ll a(x, y)\}$$

Since $k = \bigvee_{f_{n,\epsilon} \ll k} f_{n,\epsilon}$, for $n \geq 0$ and $\epsilon \in [0, 1]$, it is enough to consider the open balls of the form

$$B(f_{n,\epsilon}, x) = \{y \in X : \epsilon < a(x, y)(n)\},$$

or, as Flagg does,

$$B_\epsilon(x) = \{y \in X : 1 - \epsilon < a(x, y)(\epsilon)\},$$

considering $k = \bigvee\{f_{\epsilon,1-\epsilon} : f_{\epsilon,1-\epsilon} \ll k\}$.

Our next goal is to establish the conditions under which the topology induced by the closure operator introduced in Section 3.1, satisfying (v) of Proposition 3.4, and the one considered in [Fla97] are equivalent. To achieve this, we will show that the closure of a subset of a V-category (X, a) is the same in both topologies.
Denote the topology due to Flagg by τ_1 and the one defined by the closure operator by τ_2. As we do not require the symmetry of the structure a in the V-category (X, a), consider, for any $x \in X$ and $u \ll k$, the following adjustment to the definition of open ball:

$$B_u(x) = B(x, u) \cap B(u, x)$$

where $B(x, u) = \{y \in X : u \ll a(x, y)\}$ e $B(u, x) = \{y \in X : u \ll a(y, x)\}$.
Let $A \subseteq X$ and $x \in \overline{A}$ in τ_1. We know that, for all $u \ll k$,

$$\begin{aligned}
&B_u(x) \cap A \neq \emptyset \\
\Leftrightarrow &\exists y \in A : u \ll a(x, y) \quad \text{and} \quad u \ll a(y, x) \\
\Rightarrow &\exists y \in A : u \otimes u \ll a(x, y) \otimes a(y, x) \\
\Rightarrow &u \otimes u \ll \bigvee_{y \in A} a(x, y) \otimes a(y, x).
\end{aligned}$$

Thus

$$\bigvee_{u \ll k} u \otimes u \leq \bigvee_{y \in A} a(x,y) \otimes a(y,x).$$

If $k = \bigvee_{u \ll k} u \otimes u$, then

$$\begin{aligned} & k \leq \bigvee_{y \in A} a(x,y) \otimes a(y,x) \\ \Leftrightarrow & x \in \overline{A}, \text{ with } \overline{A} \text{ the closure of } A \text{ in } \tau_2 \end{aligned}$$

Now consider $x \in \overline{A}$, in τ_2. Thus,

$$\begin{aligned} & k \leq \bigvee_{y \in A} a(x,y) \otimes a(y,x) \\ \Rightarrow & \forall u \ll k, u \ll \bigvee_{y \in A} a(x,y) \otimes a(y,x) \\ \Rightarrow & \forall u \ll k, \exists y \in A : u \leq a(x,y) \otimes a(y,x). \end{aligned}$$

Assuming that $k = \top$ and that, for any $u \ll k$, there is $u' \ll k$ such that $u \ll u'$,

$$\begin{aligned} & \forall u \ll k, \exists y \in A : u \leq a(x,y) \text{ and } u \leq a(y,x) \\ \Leftrightarrow & \forall u \ll k, \exists y \in A : u \ll a(x,y) \text{ and } u \ll a(y,x) \\ \Leftrightarrow & x \in \overline{A} \text{ in } \tau_1. \end{aligned}$$

Thus,

Theorem 3.49. *If* V *is a quantale with* $k = \top$ *satisfying for all* $u, v \in \mathsf{V}$*, if* $k \leq u \vee v$ *then* $k \leq u$ *or* $k \leq v$ *and* $k = \bigvee_{u \ll k} u \otimes u$ *then the topologies* τ_1 *and* τ_2 *coincide.*

By Remark 3.6,

Corollary 3.50. *If* V *is a quantale with* $k = \top$, $A = \{u \in \mathsf{V} : u \ll k\}$ *is directed and* $\bigvee A = k$ *then the topologies* τ_1 *and* τ_2 *coincide.*

Examples 3.51. The conditions of Theorem 3.49 are satisfied in the quantales $\mathsf{V} = 2$, $\mathsf{V} = [0, +\infty]$ and for $\mathsf{V} = \Delta$. Hence, both topologies coincide in an ordered set, in a metric space and in a probabilistic metric space.

The origin of the notion of Cauchy sequence is eventually connected to the notion of metric. A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) is a *Cauchy sequence* if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall m, n \geq N, d(x_m, x_n) \leq \epsilon.$$

The generalisation of this definition to any V-category gives an equivalent description of a Cauchy sequence.

Theorem 3.52. *In a quantale V where $k = \top$ and $k = \bigvee_{u \ll k} u$, a sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if*

$$\forall u \ll k, \exists N \in \mathbb{N} : \forall m, n \geq N, u \leq a(x_m, x_n).$$

Proof. By definition, a sequence $(x_n)_{n\in\mathbb{N}}$ in a V-category (X, a) is Cauchy if

$$\begin{aligned}
& k \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{m,n\geq N} a(x_m, x_n) \\
\Leftrightarrow & \forall u \ll k, u \ll \bigvee_{N\in\mathbb{N}} \bigwedge_{m,n\geq N} a(x_m, x_n) \\
\Leftrightarrow & \forall u \ll k, \exists N \in \mathbb{N} : u \leq \bigwedge_{m,n\geq N} a(x_m, x_n) \\
\Leftrightarrow & \forall u \ll k, \exists N \in \mathbb{N} : \forall m, n \geq N, u \leq a(x_m, x_n).
\end{aligned}$$

□

Example 3.53. For $\mathsf{V} = \Delta$, we obtain the following characterisation of a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in a Δ-category (X, a):

$$\forall f \ll k, \exists N \in \mathbb{N} : \forall m, p \geq N, f \leq a(x_m, x_p).$$

Since, for any $f \in \Delta$, $f = \bigvee_{f_{n,\epsilon} \ll f} f_{n,\epsilon}$ then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if

$$\begin{aligned}
& \forall f \ll k, \exists N \in \mathbb{N} : \forall m, p \geq N, \bigvee_{f_{n,\epsilon} \ll f} f_{n,\epsilon} \leq a(x_m, x_p) \\
\Leftrightarrow & \forall \epsilon \in]0, 1[, \forall n > 0, \exists N \in \mathbb{N} : \forall m, p \geq N, f_{n,\epsilon} \ll a(x_m, x_p) \\
\Leftrightarrow & \forall \epsilon \in]0, 1[, \forall n > 0, \exists N \in \mathbb{N} : \forall m, p \geq N, \epsilon \leq a(x_m, x_p)(n),
\end{aligned}$$

and this last formulation is the one presented by Gregori and Romaguera [GR02].

Following the same methodology, the notion of convergence introduced in Definition 3.24 will be compared with the one deduced from the usual notion in metric spaces, considering that the distance is given by the structure of the V-category involved.

Theorem 3.54. *Let V be a quantale with $k = \top$ and $k = \bigvee_{u \ll k} u \otimes u$. A sequence $s = (x_n)_{n\in\mathbb{N}}$ in a V-category (X, a) converges to $x \in X$ if and only if*

$$\forall u \ll k, \exists N \in \mathbb{N} : \forall n \geq N, u \leq a(x_n, x) \text{ and } u \leq a(x, x_n).$$

Proof. Let $s \longrightarrow x$. Then, for any infinite subset M of $\mathbb{N}$,

$$\begin{aligned}
&k \leq \bigvee_{m\in M} a(x,x_m)\otimes a(x_m,x) \leq \bigvee_{m\in\mathbb{N}} a(x,x_m)\otimes a(x_m,x)\\
\Leftrightarrow &\forall u \ll k, u \ll \bigvee_{m\in\mathbb{N}} a(x,x_m)\otimes a(x_m,x)\\
\Leftrightarrow &\forall u \ll k, \exists m\in\mathbb{N} : u \leq a(x,x_m)\otimes a(x_m,x)\\
\Rightarrow &\forall u \ll k, \exists m\in\mathbb{N} : u \leq a(x,x_m) \text{ and } u \leq a(x_m,x).
\end{aligned}$$

Since the previous development is valid for any subsequence of s one concludes that

$$\forall u \ll k, \exists m\in\mathbb{N} : \forall n \geq m, u \leq a(x,x_n) \text{ and } u \leq a(x_n,x).$$

Now if $M \subseteq \mathbb{N}$ is infinite, given $n \in \mathbb{N}$ there is $m \in M$ such that $n \leq m$. Therefore,

$$\begin{aligned}
&\forall u \ll k, \exists m\in M : u \leq a(x,x_m) \text{ and } u \leq a(x_m,x)\\
\Rightarrow &\forall u \ll k, \exists m\in M : u\otimes u \leq a(x,x_m)\otimes a(x_m,x)\\
\Rightarrow &\forall u \ll k, u\otimes u \leq \bigvee_{m\in M} a(x,x_m)\otimes a(x_m,x)\\
\Leftrightarrow &\bigvee_{u\ll k} u\otimes u \leq \bigvee_{m\in M} a(x,x_m)\otimes a(x_m,x)\\
\Leftrightarrow &k \leq \bigvee_{m\in M} a(x,x_m)\otimes a(x_m,x).
\end{aligned}$$

□

Note that in a quantale V where $k=\top$, we have $k=\bigvee_{u\ll k} u$ whenever $k=\bigvee_{u\ll k} u\otimes u$.

Example 3.55. Let $\mathsf{V}=\Delta$. A sequence $s=(x_n)_{n\in\mathbb{N}}$ in a Δ-category (X,a) converges to $x\in X$ if and only if

$$\forall f \ll k, \exists N\in\mathbb{N} : \forall m \geq N, f \leq a(x_m,x) \text{ and } f \leq a(x,x_m).$$

Since $f=\bigvee_{f_{n,\epsilon}\ll f} f_{n,\epsilon}$, with $n\in[0,+\infty]$ and $\epsilon\in[0,1]$,

$$\begin{aligned}
&\forall \epsilon \in]0,1[, \forall n>0, \exists N\in\mathbb{N} : \forall m\geq N, f_{n,\epsilon} \ll a(x_m,x) \text{ and } f_{n,\epsilon} \ll a(x,x_m)\\
\Leftrightarrow\quad &\forall \epsilon \in]0,1[, \forall n>0, \exists N\in\mathbb{N} : \forall m\geq N, [\epsilon \leq a(x_m,x)(n) \text{ and } \epsilon \leq a(x,x_m)(n)].
\end{aligned}$$

An equivalent description of convergence in a Δ-category (X,a) is given in [GR02]: $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if $a(x_n,x)(t) \longrightarrow 1$, when $n \longrightarrow +\infty$.

Remark 3.56. This section has allowed us to assess the extent to which we obtain similar results to those of Flagg, using a different approach. Using the conceptual power of adjunctions, it is concluded that such a demanding background is not required. For example, it is not necessary the quantale to be completely distributive and, instead of a topological closure operator, it is enough to consider a closure operator (and impose some conditions on the quantale) to obtain equivalence between the main concepts discussed, in particular as regards the completeness of a V-category, as discussed in Section 3.4.

Chapter 4

Injectivity and exponentiation in V-categories

After establishing the equivalence between L-injectivity and L-completeness, now it is intended to relate injectivity and exponentiability. Starting from the assumptions and concepts set out in [CH09], another kind of completeness is considered and the formal ball model (see [KW11]) is analysed.

4.1 Exponentiable V-categories

A V-category (X, a) is *exponentiable* if the functor $X \times - : \mathsf{V\text{-}Cat} \to \mathsf{V\text{-}Cat}$ has a right adjoint $[-, X] : \mathsf{V\text{-}Cat} \to \mathsf{V\text{-}Cat}$. In that case, for any other V-category Z, there is a natural bijection

$$\mathsf{V\text{-}Cat}(X \times Y, Z) \simeq \mathsf{V\text{-}Cat}(Y, [Z, X])$$

for all V-category Y. Suppose that X is exponentiable and let $Y = 1$. Since

$$\mathsf{V\text{-}Cat}(X \times 1, Z) \simeq \mathsf{V\text{-}Cat}(1, [Z, X]) \simeq [Z, X]$$

the underlying set of the V-category $([Z, X], d)$ is $\mathsf{V\text{-}Cat}(X \times 1, Z)$. Note that $X \times 1 = (X, a')$ where $a' = a \wedge k$. Hence if $k = \top$, $X \times 1 = (X, a)$ and $([Z, X], d)$ is exactly the set of V-functors from X to Z.

The structure d in the V-category $[Z, X]$ is the biggest that makes the evaluation map

$$\begin{aligned} ev : X \times \mathsf{V\text{-}Cat}(X \times 1, Z) &\to Z \\ (x, f) &\mapsto f(x) \end{aligned}$$

a V-functor, that is, for any $x, x' \in X$ and any $h, h' \in \mathsf{V}\text{-}\mathsf{Cat}(X \times 1, Z)$,

$$a(x, x') \wedge d(h, h') \leq c(h(x), h(x')).$$

In [CH06, CHS09] it is shown:

Theorem 4.1. *In a quantale completely distributive* V*, a* V*-category* (X, a) *is exponentiable if and only if, for all* $x_0, x_2 \in X$ *and* $u, v \in \mathsf{V}$*,*

$$\bigvee_{x \in X} (a(x_0, x) \wedge u) \otimes (a(x, x_2) \wedge v) \geq a(x_0, x_2) \wedge (u \otimes v).$$

Example 4.2. The characterisation of exponentiable metric spaces is also given by [CH06]. In the quantale $[0, +\infty]$, a metric space (X, a) is exponentiable if and only if for all $x_0, x_2 \in X$ and $u, v \in [0, +\infty]$ such that $u + v = a(x_0, x_2)$,

$$\forall \epsilon > 0, \exists x \in X : a(x_0, x) < u + \epsilon \text{ and } a(x, x_2) < v + \epsilon.$$

For any category (X, a), the functor $X \otimes -$ has a right adjoint $(-)^X$ as it was seen in Section 2.2. Therefore, any V-category is "tensor-exponentiable".

Theorem 4.3. *If* V *is completely distributive the following statements are equivalent:*

i) For all $u, v, w \in \mathsf{V}$*,* $w \wedge (u \otimes v) = \bigvee \{u' \otimes v' : u' \leq u, v' \leq v, u' \otimes v' \leq w\}$*;*

ii) Every injective V*-category* X *is exponentiable;*

iii) The V*-category* V *is exponentiable.*

Proof. Suppose that the quantale V satisfies the condition in (i) and that (X, a) is an injective V-category. Let $x_0, x_2 \in X$, $u, v \in \mathsf{V}$, $w = a(x_0, x_2)$ and $u', v' \in \mathsf{V}$ such that $u' \leq u$, $v' \leq v$ and $u' \otimes v' \leq w$. To prove that X is exponentiable consider the V-categories (Y, b) and (Z, c) with $Y = \{0, 2\}$, $Z = \{0, 1, 2\}$, and

$$b(0,2) = c(0,2) = w, \qquad c(0,1) = u', \qquad c(1,2) = v'.$$

and $b(x, x) = c(x, x) = k$ and $b(x, y) = c(x, y) = \perp$ in all other cases. Furthermore, consider also the V-functors $i : Y \to Z$ and $f : Y \to X$ with $i(j) = j$ and $f(j) = x_j$, for $j = 0, 2$. The injectivity of X guarantees the existence of a V-functor $g : Z \to X$ such that $g \cdot i \simeq f$. Then, for $x = g(1)$, one has $u' \leq a(x_0, x)$ and $v' \leq a(x, x_2)$. Therefore, for such $x \in X$,

$$u' \leq a(x_0, x) \wedge u \quad \text{and} \quad v' \leq a(x, x_2) \wedge v.$$

Thus

$$u' \otimes v' \leq \bigvee_{x \in X} (a(x_0, x) \wedge u) \otimes (a(x, x_2) \wedge v),$$

and

$$\bigvee \{u' \otimes v' : u' \leq u, v' \leq v, u' \otimes v' \leq w\} \leq \bigvee_{x \in X} (a(x_0, x) \wedge u) \otimes (a(x, x_2) \wedge v).$$

Finally,

$$a(x_0, x_2) \wedge (u \otimes v) \leq \bigvee_{x \in X} (a(x_0, x) \wedge u) \otimes (a(x, x_2) \wedge v),$$

that is, X is exponentiable.

If every injective V-category is exponentiable then every quantale V is exponentiable since, by Lemma 3.41, $(\mathsf{V}, \mathsf{hom})$ is an injective V-category.

To prove the implication (iii)⇒(i) suppose that $(\mathsf{V}, \mathsf{hom})$ is exponentiable. Thus, for $w, u, v \in \mathsf{V}$:

$$\bigvee_{x \in X} (\mathsf{hom}(k, x) \wedge u) \otimes (\mathsf{hom}(x, w) \wedge v) = \mathsf{hom}(k, w) \wedge (u \otimes v) = w \wedge (u \otimes v)$$

Since $\mathsf{hom}(k, x) \wedge u \leq u$, $\mathsf{hom}(x, w) \wedge v \leq v$ and $(\mathsf{hom}(k, x) \wedge u) \otimes (\mathsf{hom}(x, w) \wedge v) \leq w$ one obtains (i). □

Corollary 4.4. *For a completely distributive quantale* V *satisfying the condition* (i) *of Theorem 4.3, the full subcategory of* V-Cat *whose objects are injective* V*-categories is Cartesian closed.*

Proof. Any Cartesian product of injective V-categories is injective. In fact, let X_1 e X_2 be injective V-categories, $f : Y \to X_1 \times X_2$ and $i : Y \to Z$ V-functors with i fully faithful.

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "f"'] & Z \arrow[d, dashed] \arrow[dr, dashed] & \\ & X_1 \times X_2 \arrow[r, "\pi_i"] & X_i \end{tikzcd}$$

For $i = 1, 2$, since X_i is injective then there is $g_i : Z \to X_i$ such that $g_i \cdot i \simeq \pi_i \cdot f$, and, by the universal property of the Cartesian product, there is $g : Z \to X_1 \times X_2$ such that $\pi_i \cdot g \simeq g_i$. Then such g verifies $g \cdot i \simeq f$.

Let X and Y be injective V-categories, $i : A \to B$ and $f : A \to Y^X$ V-functors with i fully faithful. Since V satisfies (i) of Theorem 4.3, X is exponentiable and we can conclude that there is a V-functor $\hat{f} : X \times A \to Y$. Thus, Lemma 2.14 and the injectivity of Y guarantee the existence of a V-functor $\hat{g} : X \times B \to Y$ such that

$\hat{g} \cdot (1_X \times i) \simeq \hat{f}$. Again by the exponentiability of X, to $\hat{g}$ corresponds g that satisfies the required condition: $g \cdot i \simeq f$. Thus Y^X is injective. □

Examples 4.5. 1. Given $u, v, w \in [0, +\infty]$ there are always $u', v' \in [0, +\infty]$ such that $\sup\{w, u + v\} = u' + v'$. This means that a injective metric space is exponentiable and, in particular, that $[0, +\infty]$ is exponentiable. Furthermore, the subcategory of Met defined by the injective metric spaces is Cartesian closed.

2. In $\mathsf{V} = \Delta$, we can guarantee that, given u, v and w, there are u' and v' such that $w \wedge (u \otimes v) = u' \otimes v'$. This means that (ii) and (iii) of Theorem 4.3 are also valid in Δ.

   In fact, since any distribution function f can be described as the supremum of those $f_{n,\epsilon} \in \Delta$ totally bellow f, it is enough to consider $u = f_{n_1,\epsilon_1}$, $v = f_{n_2,\epsilon_2}$ and $w = f_{n_3,\epsilon_3}$; then

   $$u \otimes v = f_{n_1+n_2,\epsilon_1\cdot\epsilon_2}$$

   There are several possibilities:

   - if $w \geq u \otimes v$ (that is $n_1 + n_2 \geq n_3$ and $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_3$), then $w \wedge (u \otimes v) = u \otimes v$;
   - if $w \leq u \otimes v$ (that is $n_1 + n_2 \leq n_3$ and $\epsilon_1 \cdot \epsilon_2 \geq \epsilon_3$), then $w \wedge (u \otimes v) = u \otimes v'$ where $v' = f_{n,\epsilon}$ with $n_1 + n = n_3$, $n \geq n_2$, and $\epsilon_1 \cdot \epsilon = \epsilon_3$, $\epsilon \leq \epsilon_2$;
   - if $n_1 + n_2 \leq n_3$ and $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_3$, then $w \wedge (u \otimes v) = u \otimes v'$ where $v' = f_{n,\epsilon}$ with $n_1 + n = n_3$, $n \geq n_2$, and $\epsilon = \epsilon_2$;
   - if $n_1 + n_2 \geq n_3$ and $\epsilon_1 \cdot \epsilon_2 \geq \epsilon_3$, then $w \wedge (u \otimes v) = u \otimes v'$ where $v' = f_{n,\epsilon}$ with $n = n_2$, and $\epsilon_1 \cdot \epsilon = \epsilon_3$, $\epsilon \leq \epsilon_2$.

   Therefore, every injective probabilistic metric space is exponentiable and, in particular, Δ is exponentiable. Furthermore, the subcategory of ProbMet defined by the injective probabilistic metric spaces is Cartesian closed.

As seen in Example 4.5 (1), the quantale $[0, +\infty]$ satisfies (i) of Theorem 4.3. Hence the full subcategory of Met whose objects are injective metric spaces is Cartesian closed. In [Wag94] is stated that the largest Cartesian closed subcategory of Met is the category of ultrametric spaces, UMet. However, the metric space $([0, +\infty], \ominus)$ is injective but it is not ultrametric.

4.2 Injectivity and the formal ball model

As we have seen in the proof of Theorem 4.3, it is not required such a strong condition as injectivity in (ii) to have exponentiability. In fact the injectivity is only necessary in a particular situation. Let

$$\Phi X = \{\psi : X \longrightarrow\!\!\!\!\!\!\!\circ\;\, 1 : \psi = u \cdot x^*, x \in X, u \in \mathsf{V}\}.$$

The elements of ΦX are those V-distributors $\psi : X \longrightarrow\!\!\!\!\!\!\!\circ\;\, 1$ for which there is $u \in \mathsf{V}$ and $x \in X$ such that $\psi(x') = u \otimes a(x', x)$, for all $x' \in X$. This set equipped with the structure of the V-category $(\hat{X}, \hat{a})$ is also a V-category, where, by Lemma 2.5,

$$\hat{a}(u \cdot x^*, v \cdot y^*) = \mathsf{hom}(u, a(x, y) \otimes v),$$

for all $u, v \in \mathsf{V}$ and all $x, y \in X$.

Definition 4.6. A V-functor $f : (Y, b) \to (Z, c)$ is Φ-*dense* if, for all $z \in Z$, $z^* \cdot f_* \in \Phi Y$, or equivalently, if, for all $z \in Z$, there is $u \in \mathsf{V}$ and $y \in Y$ such that $z^* \cdot f_* = u \cdot y^*$. In pointwise notation,

$$c(f(y'), z) = u \otimes b(y', y),$$

for all $y' \in Y$.

Note that the restriction of the Yoneda V-functor, $\mathcal{Y}_X : X \to \Phi X$, is Φ-dense because, for all $u \cdot x^* \in \Phi X$ and all $x' \in X$,

$$\hat{a}(a(-, x'), u \cdot x^*) = u \cdot x^*(x') = u \otimes a(x', x).$$

Definition 4.7. A V-category (X, a) is Φ-*injective* whenever, for all V-categories (Y, b) and (Z, c) and for all V-functors $i : Y \to Z$ and $f : Y \to X$ with i Φ-dense, there is a V-functor $g : Y \to X$ such that $g \cdot i \simeq f$.

Definition 4.8. A V-category (X, a) is Φ-*cocomplete* if $\mathcal{Y}_X : X \to \Phi X$ has a left adjoint, $S_X : \Phi X \to X$.

Theorem 4.9. *A V-category (X, a) is Φ-cocomplete if and only if, for every $u \in \mathsf{V}$ and $x \in X$, there is $u \otimes x \in X$ such that*

$$a(u \otimes x, y) = \mathsf{hom}(u, a(x, y)),$$

for all $y \in X$.

Proof. A V-category (X, a) is Φ-cocomplete if and only if there is a map $S_X : \Phi X \to X$ that is left adjoint to $\mathcal{Y}_X : X \to \Phi X$. Thus X is Φ-cocomplete if and only if, for all $u \in \mathsf{V}$ and all $x \in X$, there is $u \otimes x \in X$, where $u \otimes x$ is a representation for $S_X(u \cdot x^*)$, such that,

$$\begin{aligned}
&a(S_X(u\cdot x^*),y)=\hat{a}(u\cdot x^*,\mathcal{Y}_X(y))\\
\Leftrightarrow &a(u\otimes x,y)=\hat{a}(u\cdot x^*,y^*)\\
\Leftrightarrow &a(u\otimes x,y)=\hat{a}(u\cdot a(-,x),a(-,y))\\
\Leftrightarrow &a(u\otimes x,y)=\bigwedge_{x'\in X}\mathsf{hom}(u\otimes a(x',x),a(x',y))\\
\Leftrightarrow &a(u\otimes x,y)=\mathsf{hom}(u,a(x,y)).
\end{aligned}$$ □

Proposition 4.10. *A V-category (X,a) is Φ-injective if and only if it is Φ-cocomplete.*

Proof. By adapting the proof of Proposition 3.40 to the new set of Φ-notions one obtains the desired result. □

One can think of different classes Φ of distributors that induce different notions of injectivity and of completeness. In [CH09] was studied cocompleteness with respect to a class of distributors (see also [AK88, KL00, KS05]). In the present Subsection we are working on the particular case of V-distributors of the form $u\cdot x^*$, for $u\in\mathsf{V}$ and $x\in X$.

Examples 4.11. Let X_1 e X_2 be Φ-injectives V-categories. By the adapting the first part of the proof of Corollary 4.4 we conclude:

Lemma 4.12. *The Cartesian product of Φ-injective V-categories is Φ-injective.*

However, when we consider the tensor product, this generalization is not valid.
In fact, let (X,a) be a Φ-cocomplete metric space. This means that $\mathcal{Y}_X$ has a left adjoint $S_X:\Phi X\to X$, that takes an element $u\cdot x^*$ of ΦX to an element of X denoted by $u\otimes x$. In the quantale $\mathsf{V}=[0,+\infty]$ the adjunction $S_X\dashv\mathcal{Y}_X$ is characterised by

$$a(u\otimes x,y)=a(x,y)\ominus u,$$

for all $u\in\mathsf{V}$ and $x,y\in X$. If also (Y,b) is a Φ-cocomplete metric space then $X\otimes Y$ is Φ-cocomplete if there is $S_{X\otimes Y}:\Phi(X\otimes Y)\to X\otimes Y$ such that, for all $x,x'\in X$, all $y,y'\in Y$, all $n\in[0,+\infty]$ and, considering $S_{X\otimes Y}(n\cdot(x,y)^*)=(x_0,y_0)$,

$$(a\otimes b)((x_0,y_0),(x',y'))=(a\otimes b)((x,y),(x',y'))\ominus n,$$

or, equivalently,

$$a(x_0,x')+b(y_0,y')=(a(x,x')+b(y,y'))\ominus n.$$

In particular, for $X = Y = [0, +\infty]$, $S_{X \otimes Y}$ must satisfy,

$$(x' \ominus x_0) + (y' \ominus y_0) = ((x' \ominus x) + (y' \ominus y)) \ominus n,$$

for all $(x, y), (x', y') \in X \times Y$ and all $n \in [0, +\infty]$. Since this equality must be valid for every elements of $[0, +\infty]$ and of $X \times Y$, consider $n = 1$ and $(x, y) = (0, 0)$; hence, there must exist $x_0, y_0 \in [0, +\infty]$ such that, for all $x', y' \in [0, +\infty]$,

$$(x' \ominus x_0) + (y' \ominus y_0) = (x' + y') \ominus 1.$$

Giving values to x' and y',

1. For $x' = 1$ and $y' = 0$: $1 - x_0 = 0 \Rightarrow x_0 \geq 1$;
2. For $x' = 0$ and $y' = 1$: $1 - y_0 = 0 \Rightarrow y_0 \geq 1$;
3. For $x' = 1$ and $y' = 1$: $(1 - x_0) + (1 - y_0) = 1$ which is impossible by (1) and (2).

Thus the Yoneda V-functor on $[0, +\infty] \otimes [0, +\infty]$ has no left adjoint and, consequently, this is not a Φ-cocomplete $[0, +\infty]$-category.

Our next goal is to study the Φ-cocompleteness of Y^X, where $(-)^X$ denotes the right adjoint to $X \otimes (-)$ and X and Y are Φ-cocomplete V-categories. Consider a Φ-dense V-functor $i : Z \to W$ and a V-functor $f : Z \to Y^X$.

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "f"'] & W \arrow[d, dotted] \\ & Y^X \end{tikzcd}$$

Since $X \otimes (-)$ has right adjoint $(-)^X$ then there is a V-functor $\hat{f} : X \otimes Z \to Y$ such that $\hat{f}(x, z) = f(z)(x)$.

Lemma 4.13. *The V-functor $1_X \otimes i : X \otimes Z \to X \otimes W$ is Φ-dense whenever $i : Z \to W$ is Φ-dense.*

Proof. Denote the structure on X, Z and W by a, c and d, respectively. By hypothesis, for all $w \in W$, there is $u \in \mathsf{V}$ and $z \in Z$, such that $w^* \cdot i_* = u \cdot z^*$, ie,

$$d(i(z'), w) = u \otimes c(z', z),$$

for all $z' \in Z$. Given $(y,z') \in X \times Z$ and $(x,w) \in X \times W$,

$$\begin{aligned}(a \otimes d)(1_X \otimes i(y,z'),(x,w)) &= (a \otimes d)((y,i(z')),(x,w)) \\ &= a(y,x) \otimes d(i(z'),w) \\ &= a(y,x) \otimes u \otimes c(z',z) \\ &= u \otimes (a \otimes c)((y,z'),(x,z)).\end{aligned}$$

Thus $1_X \otimes i$ is Φ-dense. $\square$

Returning to the analysis we were developing before introducing the previous lemma, since Y is Φ-injective we conclude that there is V-functor $\hat{g} : X \otimes W \to Y$ such that $\hat{g} \cdot (1_X \otimes i) = \hat{f}$.

$$\begin{tikzcd} X \otimes Z \arrow[r, "1_X \otimes i"] \arrow[dr, "\hat{f}"'] & X \otimes W \arrow[d, dotted, "\hat{g}"] \\ & Y \end{tikzcd}$$

Then there is also a V-functor $g : W \to Y^X$ such that $g(w)(x) = g(x,w)$. Since $g \cdot i(z)(x) = \hat{g}(x,i(z)) = \hat{f}(x,z) = f(z)(x)$ then Y^X is Φ-injective.

The V-functor $i : Y \to Z$ used in the proof of Theorem 4.3 ((i) $\Rightarrow$ (ii)) is Φ-dense. In fact, for all $y \in Y$, $c(y,0) = k \otimes b(y,0)$, $c(y,1) = u' \otimes c(y,0)$ and $c(y,2) = k \otimes b(y,2)$. Therefore, we can state that (i) and the Φ-injectivity of X imply the exponentiability of X. Thus, using a similar reasoning,

Theorem 4.14. *If* V *is completely distributive, the following statements are equivalent:*

i) For all $u,v,w \in \mathsf{V}$, $w \wedge (u \otimes v) = \bigvee\{u' \otimes v' : u' \leq u, v' \leq v, u' \otimes v' \leq w\}$*;*

ii) Every Φ*-injective* V*-category* X *is exponentiable;*

iii) V *is exponentiable.*

Example 4.15. The formal ball model [KW11] over a V-category (X,a) is the set

$$BX = \{(x,u) : x \in X, u \in \mathsf{V}\}.$$

This set is also a V-category with the structure induced by the structure on ΦX:

$$BX((x,u),(x',u')) = \mathsf{hom}(u, a(x,x') \otimes u'),$$

for all $x, x' \in X$ and all $u, u' \in \mathsf{V}$. Furthermore, it is an ordered set; for $\psi, \psi' \in BX$, with $\psi = (x, u)$ and $\psi' = (x', u')$, let

$$\begin{aligned} \psi \leq \psi' &\Leftrightarrow (x, u) \leq (x', u') \\ &\Leftrightarrow k \leq BX((x, u), (x', u')) \\ &\Leftrightarrow k \leq \mathsf{hom}(u, a(x, x') \otimes u') \\ &\Leftrightarrow u \leq a(x, x') \otimes u'. \end{aligned}$$

This example highlights the difference and importance of the two concepts of completeness discussed, as BX is Φ-injective but it is not injective.
Suppose that k is the top element in the quantale V. To prove that BX is Φ-injective we will show that BX is Φ-cocomplete. Let $S_{BX} : \Phi BX \to BX$ such that, for $u, v \in \mathsf{V}$ and $x \in X$, $S_{BX}(u \cdot (x, v)^*) = (x, u \otimes v)$. Then we have $S_{BX} \dashv \mathcal{Y}_{BX}$ precisely if

$$\begin{aligned} & BX((x, u \otimes v), (x', v')) = \mathsf{hom}(u, BX((x, v), (x', v')) \\ \Leftrightarrow\quad & \mathsf{hom}(u \otimes v, a(x, x') \otimes v') = \mathsf{hom}(u, \mathsf{hom}(v, a(x, x') \otimes v')). \end{aligned}$$

Both inequalities are obvious:

$$\begin{aligned} & \mathsf{hom}(u \otimes v, a(x, x') \otimes v') \leq \mathsf{hom}(u, \mathsf{hom}(v, a(x, x') \otimes v')) \\ \Leftrightarrow\quad & u \otimes \mathsf{hom}(u \otimes v, a(x, x') \otimes v') \leq \mathsf{hom}(v, a(x, x') \otimes v') \\ \Leftrightarrow\quad & v \otimes u \otimes \mathsf{hom}(u \otimes v, a(x, x') \otimes v') \leq a(x, x') \otimes v' \\ \Leftrightarrow\quad & a(x, x') \otimes v' \leq a(x, x') \otimes v', \end{aligned}$$

and

$$\begin{aligned} & u \otimes \mathsf{hom}(u, \mathsf{hom}(v, a(x, x') \otimes v')) \leq \mathsf{hom}(v, a(x, x') \otimes v') \\ \Leftrightarrow\quad & v \otimes u \otimes \mathsf{hom}(u, \mathsf{hom}(v, a(x, x') \otimes v')) \leq a(x, x') \otimes v' \\ \Leftrightarrow\quad & \mathsf{hom}(u, \mathsf{hom}(v, a(x, x') \otimes v')) \leq \mathsf{hom}(u \otimes v, a(x, x') \otimes v'). \end{aligned}$$

To verify that the V-category BX is not injective note that the functor $G : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{Ord}$, that takes a V-category (X, a) to the ordered set $(X \leq)$ where $x \leq y$ if $k \leq a(x, y)$, has a left adjoint $F : \mathsf{Ord} \to \mathsf{V}\text{-}\mathsf{Cat}$ where $a(x, y) = k$ if $x \leq y$ and else $a(x, y) = \bot$. Since F is the functor induced by the morphism of quantales $I : \mathsf{2} \to \mathsf{V}$ then, by Lemma 3.46, F preserves fully faithful and fully dense $\mathsf{2}$-functors (monotone maps). Thus, by Theorem 3.45, G preserves injectivity. Therefore, if (X, a) is an injective V-category also GX is injective in Ord, which is equivalent to say that GX is a complete ordered

set (see [BB67]).
Hence it is enough to show that there are two formal balls with no supremum. Let $\psi_0 = (x_0, k)$, $\psi_1 = (x_1, k)$ and $\psi = (x, u)$ be elements of BX such that $x_0 \neq x_1$, and suppose that $\psi_0, \psi_1 \leq \psi$; then

$$\begin{aligned} & k \leq a(x_0, x) \quad \text{ and } \quad k \leq a(x_1, x) \\ \Leftrightarrow & x_0 \leq x \quad \text{ and } \quad x_1 \leq x. \end{aligned}$$

For

$$a(x, y) = \begin{cases} k & \text{if } x = y \\ \perp & \text{else} \end{cases}$$

one has $\psi_0 \leq \psi$ if $x_0 = x$ and $\psi_1 \leq \psi$ if $x_1 = x$, but $x_0 \neq x_1$. Therefore the set $\{\psi_0, \psi_1\}$ has no supremum.

Index

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